Covariance Matrices by Multiple Scattering and Energy Loss

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Abstract

We show derivations of the covariance matrix elements due to multiple scattering and energy loss. They are to be used in the Kalman filtering subroutines (modules).

1. **Introduction**

Since the covariance matrices due to multiple scattering and energy loss, suitable for our implementation, were not found in the literature, we show derivation of the formulae in this write up in the hope that bugs don't get into our later coding.

The state vector, P, is defined as in Maher's notation; i.e., we use

$$P = (x, y, R^{-1}, t, \phi) \tag{1}$$

to describe the helix trajectory in the Kalman filter. x, y, z is the usual Cartesian coordinates with the origin in the center of our spectrometer. The solenoidal magnetic field points in the positive z direction. x (or y, as you like. It doesn't matter) is upward. R is the radius of curvature of the helix track projected on the $x - y$ plane. t is $\tan \lambda$, where λ is the so called dip angle, which measures how much the track dips out of of the chamber plane. Physics-wise, t is the ratio of longitudinal to transverse momentum: $t = p_z/p_t$. R, t, and particle momentum p are related according to the following well know formula,

$$p = \frac{c \cdot B}{q} \cdot R \cdot \sqrt{1 + t^2}, \tag{2}$$

where c, B is the speed of light in vacuum and magnitude of the magnetic field, respectively. Finally, ϕ is the azimuthal angle measured from x. Note that with the definition of ϕ, R is signed in such a way that $R\phi$ goes in the right direction. Fig. 1 draws an electron trajectory in our coordinate system.

We'll have to work out the track model first before elaborating on covariance matrices.

2. **Track Model**

The helical trajectory of the charged particle in our coordinate system advances

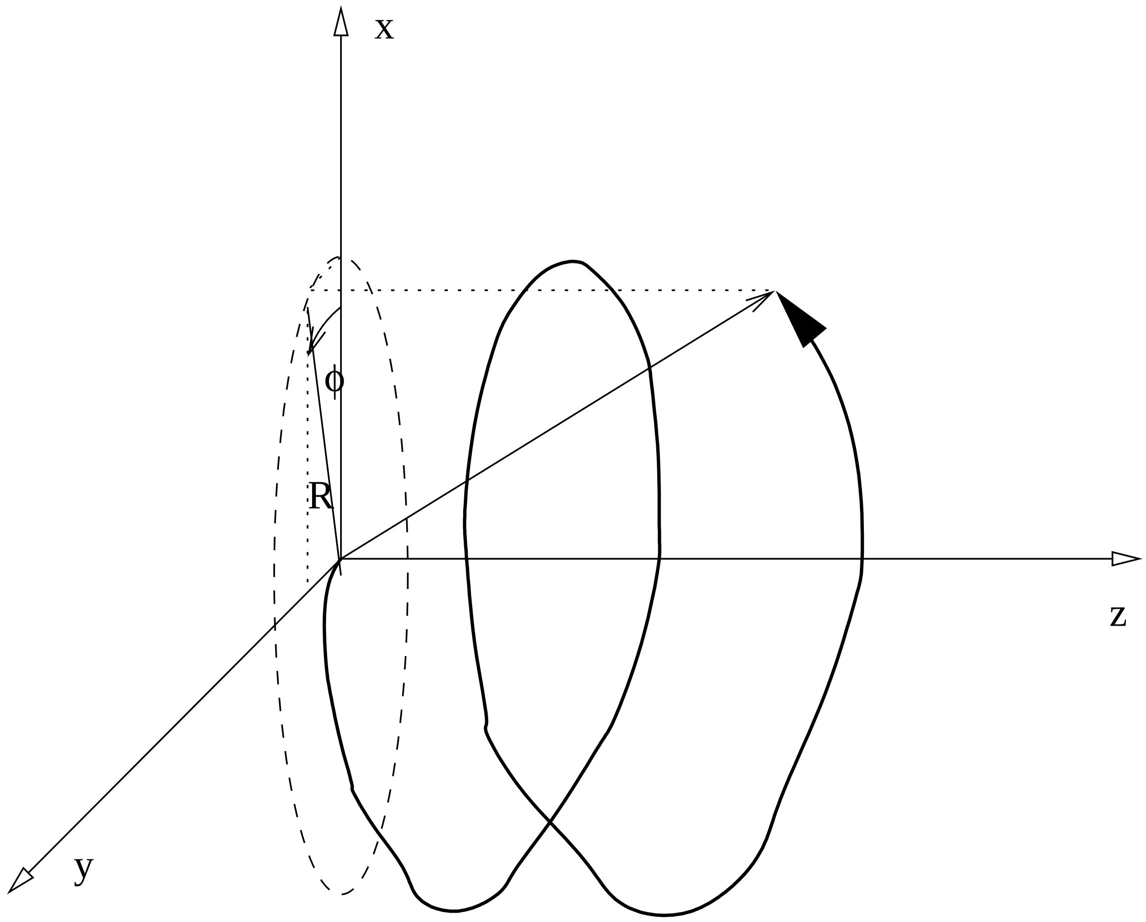

Figure 1: A helical trajectory of the electron. The solenoidal field is in the positive z direction. R is the radius of curvature of the helix projected on the $x-y$ plane. ϕ is the azimuthal angle from the x axis.

from one chamber plane at z to the next chamber plane at z' in the following way,

$$x' = x + R' \cos\phi' - R\cos\phi \tag{3a}$$

$$y' = y + R' \sin\phi' - R\sin\phi \tag{3b}$$

$$\phi' \approx \phi + \frac{z'-z}{t} \cdot \frac{2}{R'+R} \tag{3c}$$

$$R'^{-1} \approx R^{-1} - R^{-1} \cdot \frac{\Delta E}{p} \tag{3d}$$

$$t' = t \tag{3e}$$

where ΔE is the energy loss due to ionization and will be discussed in Sec 4. If we neglect the energy loss for the moment, then $R' = R$ and eqs. (3) are straightforward. Note the approximation we made in eq. (3c). If, however, $\Delta E = 0$, eqs. (3c) and (3d) are exact.

Kalman filter is an *optimal* filter for linear systems. Although eqs. (3) are not linear, we linearize them by retaining the linear terms in the Taylor series expansions around the vicinity of P, and obtain the F matrix to be used in the Kalman scheme: $P'^T = FP^T$, where the superscript T stands for matrix transpose.

The resulting nonzero matrix elements are shown below.

$$F_{11} = \frac{\partial x'}{\partial x} = 1$$
$$F_{13} = \frac{\partial x'}{\partial R^{-1}} = -R'^2 \cdot \frac{\partial R'^{-1}}{\partial R^{-1}} \cdot \cos\phi' - R' \sin\phi' \cdot \frac{\partial \phi'}{\partial R^{-1}} + R^2 \cos\phi$$
$$F_{14} = \frac{\partial x'}{\partial t} = -R'^2 \cdot \frac{\partial R'^{-1}}{\partial t} \cdot \cos\phi' - R' \sin\phi' \cdot \frac{\partial \phi'}{\partial t}$$
$$F_{15} = \frac{\partial x'}{\partial \phi} = -R' \sin\phi' + R \sin\phi$$
$$F_{22} = \frac{\partial y'}{\partial y} = 1$$
$$F_{23} = \frac{\partial y'}{\partial R^{-1}} = -R'^2 \cdot \frac{\partial R'^{-1}}{\partial R^{-1}} \cdot \sin\phi' + R' \cos\phi' \cdot \frac{\partial \phi'}{\partial R^{-1}} + R^2 \sin\phi$$
$$F_{24} = \frac{\partial y'}{\partial t} = -R'^2 \cdot \frac{\partial R'^{-1}}{\partial t} \cdot \sin\phi' + R' \cos\phi' \cdot \frac{\partial \phi'}{\partial t}$$
$$F_{25} = \frac{\partial y'}{\partial \phi} = R' \cos\phi' - R \cos\phi$$
$$F_{33} = \frac{\partial R'^{-1}}{\partial R^{-1}} = 1 - 2 \cdot \frac{\Delta E}{p}$$
$$F_{34} = \frac{\partial R'^{-1}}{\partial t} = R^{-1} \cdot \frac{t}{1+t^2} \cdot \frac{\Delta E}{p}$$
$$F_{44} = \frac{\partial t'}{\partial t} = 1$$

$$F_{53} = \frac{\partial \phi'}{\partial R^{-1}} = \frac{z' - z}{t} \cdot \frac{2}{(R' + R)^2} \cdot (R'^2 \cdot \frac{\partial R'^{-1}}{\partial R^{-1}} + R^2)$$
$$F_{54} = \frac{\partial \phi'}{\partial t} = -\frac{z' - z}{t^2} \cdot \frac{2}{R' + R}$$
$$F_{55} = \frac{\partial \phi'}{\partial \phi} = 1$$

The F matrix updates not only the state vector, but also the covariance matrix, which is the subject for the rest of this note.

3. **Multiple Scattering**

The charged particle is deflected after elastically scattering off an atomic nucleus. Two independent angles, θ_1, θ_2, are used to describe the deflection, as illustrated in most textbooks. To see how θ_1 and θ_2 affect our helix parameters, we first setup two perpendicular planes along the track. First of all, the unit vector along the track $\hat{t}_1$ is,

$$\hat{t}_1 = (\cos\lambda \cos\phi,\ \cos\lambda \sin\phi,\ \sin\lambda). \tag{4}$$

If we define $\hat{t}_2 = \hat{t}_1 \times \hat{z}$ and $\hat{t}_3 = \hat{t}_1 \times \hat{t}_2$, then, after normalization of $\hat{t}_2$ and $\hat{t}_3$, it is clear that $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$ form a local coordinate system. The new particle direction after Coulomb scattering, $\hat{t}'_1$, becomes,

$$\hat{t}'_1 = \hat{t}_1 + \theta_2 \cdot \hat{t}_2 + \theta_1 \cdot \hat{t}_3. \tag{5}$$

Substituting eq. (4) into eq. (5) and with the constructions of $\hat{t}_2$ and $\hat{t}_3$, we have, from the third component of eq. (5),

$$\sin\lambda' = \sin\lambda - \theta_1 \cos\lambda,$$

or

$$\delta\lambda = \lambda' - \lambda \approx -\theta_1. \tag{6}$$

Similarly, from the first component of eq. (5) and again retaining only first order in θ_1 and θ_2, we have,

$$\cos\lambda' \cos\phi' = \cos\lambda \cos\phi + \theta_2 \sin\phi + \theta_1 \cos\phi \sin\lambda,$$

or

$$\delta\phi = \phi' - \phi \approx -\frac{\theta_2}{\cos\lambda}. \tag{7}$$

Multiple scattering (θ_1, θ_2) thus causes uncertainty in the parameters $t = \tan\lambda$ and ϕ, which in turn incur uncertainties in x, y, and R^{-1} through the following relationships,

$$\begin{aligned} x &= R\cos\phi \\ y &= R\sin\phi \\ R^{-1} &= \frac{cB}{q} \cdot \frac{1}{p} \cdot \sqrt{1+t^2} \end{aligned} \tag{8}$$

Working out the partial derivatives of eqs. (8) and using eqs. (6) and (7), one obtains,

$$\begin{aligned} \delta x &= xt\,\theta_1 + y\sqrt{1+t^2}\ \theta_2 \\ \delta y &= yt\,\theta_1 - x\sqrt{1+t^2}\ \theta_2 \\ \delta R^{-1} &= -R^{-1}t\ \theta_1 \end{aligned} \tag{9}$$

$$\delta t = -(1+t^2)\ \theta_1$$

$$\delta\phi = -\sqrt{1+t^2}\ \theta_2$$

Now since $cov\{\theta_i, \theta_j\} = \theta_o^2\ \delta_{ij}\ (i,j=1,2)$, one gets the covariance matrix due to multiple scattering Q^{ms},

$$Q_{ij}^{ms} = \left[\frac{\partial P_i}{\partial \theta_1}\frac{\partial P_j}{\partial \theta_1} + \frac{\partial P_i}{\partial \theta_2}\frac{\partial P_j}{\partial \theta_2}\right]\theta_o^2, \qquad (i,j=1,2,\cdots,5). \tag{10}$$

Explicitly,

$$Q^{ms} = \begin{pmatrix} R^2t^2+y^2 & -xy & -t^2\cos\phi & -xt(1+t^2) & -y(1+t^2) \\ -xy & R^2t^2+x^2 & -t^2\sin\phi & -yt(1+t^2) & x(1+t^2) \\ -t^2\cos\phi & -t^2\sin\phi & t^2(R^{-1})^2 & R^{-1}t(1+t^2) & 0 \\ -xt(1+t^2) & -yt(1+t^2) & R^{-1}t(1+t^2) & (1+t^2)^2 & 0 \\ -y(1+t^2) & x(1+t^2) & 0 & 0 & 1+t^2 \end{pmatrix}\theta_o^2$$

Note that Q_{ij}^{ms} is symmetric and θ_o^2 is the well known Rossi-Greisen formula, depending on the arc length traversed and the radiation length of the material.

4. **Energy Loss**

We consider here only energy losses due to ionization when the charged particle interacts with atomic electrons through material since it is the predominant process for incident charged particles of energy below ~100 MeV.

The mean energy loss ΔE is calculable from the Bethe-Bloch formula,

$$\Delta E = \frac{dE}{dx}\Delta x, \tag{11}$$

and Δx is estimated from the measurement. However, ΔE straggles and the distribution is described by a Landau distribution for particles traversing a thin slab. We denote the r.m.s of ΔE as δE.

Since energy is lost (or gained if we perform the Kalman filtering backwards), the magnitude of the particle momentum p decreases to p':

$$p' = \sqrt{(E + \Delta E)^2 + m^2} = \sqrt{p^2 + 2E\Delta E + \Delta E^2}. \tag{12}$$

If $\Delta E << p$, to first order and neglecting the rest mass of the electron, we arrive at eq. (3d).

Similar to the treatment of multiple scattering, uncertainty in ΔE affects the accuracy of R^{-1}, which in turn gives rise to uncertainties in x and y through eqs. (8):

$$\begin{aligned}
\delta x &= x \cdot \frac{\delta E}{p} \\
\delta y &= y \cdot \frac{\delta E}{p} \\
\delta R^{-1} &= -R^{-1} \cdot \frac{\delta E}{p} \\
\delta t &= 0 \\
\delta \phi &= 0
\end{aligned} \tag{13}$$

Finally, following eq. (10), we have the elements of the covariance matrix due to ionization energy loss, Q^{el}_{ij},

$$
Q^{el} = \begin{pmatrix} x^2 & xy & -xR^{-1} & 0 & 0 \\ xy & y^2 & -yR^{-1} & 0 & 0 \\ -xR^{-1} & -yR^{-1} & (R^{-1})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} (\frac{\delta E}{p})^2
$$

5. **Summary**

The Kalman filter technique is advantageous when multiple scattering and energy loss are taken into account in the track reconstruction. The other advantage is that only a smaller matrix (5 by 5 in our case) needs to be inverted, compared with the conventional global fit method. We derived the covariance matrices due to multiple scattering and ionization energy loss. The F matrix was also revised because of the energy loss. The total covariance matrix is then $Q^{ms} + Q^{el}$. Together with the propagation matrix F , errors (or weights) are calculated from chamber plane to chamber plane according to the Kalman procedure.

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