Optics Considerations for the Injection of a 29.8 MeV/c Surface Muon Beam into a Solenoid

Jaap Doornbos, TRIUMF

July 26, 2000

Abstract

This report discusses the optimal injection of a surface muon beam into a 22.5 kG solenoid field. Simple formulas are given which make it possible to calculate the depolarization and the beam parameters in the homogeneous field of the solenoid as function of the beam parameters before the solenoid.

1 Introduction

Experiment E614 needs a surface muon beam with a FWHM momentum width of 1.0 percent $\Delta P/P$. The beam has to be extremely parallel in the center of a 22.5 kG solenoid field, where the beam is stopped, and the spectrum of decay positrons is measured relative to the direction of the muon polarization in order to determine the Michel parameters with high precision. The muon polarization is defined relative to the direction of the axis of the solenoid. Since for surface muons the spin is exactly aligned along the direction of motion, it is necessary that the angle α between the muon direction and the solenoid axis is small. α was originally envisaged to be smaller than 15 mr, which would give a maximum depolarization of $1 - \cos\alpha = 10^{-4}$. However, this would only be possible with an unuseably low muon intensity. A workable approach is to allow for larger angles, but to make sure that the uncertainty in the average depolarization in the full α distribution is close to 1.0 10^{-4}.

When a beam enters a solenoid, it acquires in the fringe field mechanical angular momentum, which increases the angles with respect to the beam axis. Therefore, it becomes very important to find an optimum way of injecting the beam. Section 2 explains the usual representation of a beam by phase ellipses, and section 3 gives the magnetic field. The optimization of the injection into the solenoid is discussed in section 4. It turns out that it is possible to express α in the homogeneous central field, the radius r to the solenoid axis, as well as x and y as simple functions of the beam parameters before the solenoid. This is done in section 5.

The expressions are valid for all longitudinal positions in the homogeous part of the magnetic field. In particular, the expression for α is very simple and very accurate. This expression can be used to calculate the depolarization in the solenoid as function of the beam parameters before the solenoid. This is demonstrated in section 6. An illustration of the application of the equations is given in section 7 for a beam for M13. In that case, the beam is either traced directly to the center of the solenoid, or the beam is traced to the exit of the last quadrupole, after which the formulas are applied to obtain the parameters α and r. Both methods give good agreement, as can be seen in figures 8 and 9.

2 Representation of a beam by phase ellipses

A beam of particles can be described by x, y, θ and ϕ. θ is the angle between the projection of a ray on the x-z plane and the z-axis. ϕ is the angle between a ray and its projection on the x-z plane. The ray is furthermore characterized by its momentum. The movement of a particle in the x-z plane can be represented by a point of the phase space ellipse in the x-θ phase plane. See the left part of figure 1. It is a tilted ellipse, which can be characterized by the parameters θ_0, x_0 and the tilt angle. θ_0 is the maximum angle in the beam. x_0 is the intersection of the beam with the x-axis. x_0 is smaller than the maximum x in the beam.

The picture shows a converging beam, because after a drift L a new ellipse is obtained with smaller maximum x. Eventually, the ellipse becomes upright and reaches its smallest size, given by x_0. The beam has a waist. See the right part of figure 1. Then the beam diverges again. Therefore, one can characterize the beam by the tilt angle of the original ellipse or by the drift length needed to obtain a waist. The area of the ellipse is $\epsilon = \pi x_0 \theta_0$. It is called the horizontal emittance of the beam. The vertical emittance can be defined in the same way using y and ϕ in the y-ϕ phase plane. The emittance of the beam is a conserved quantity. The area enclosed by the tilted ellipse is equal to the area enclosed by the upright ellipse. In the latter

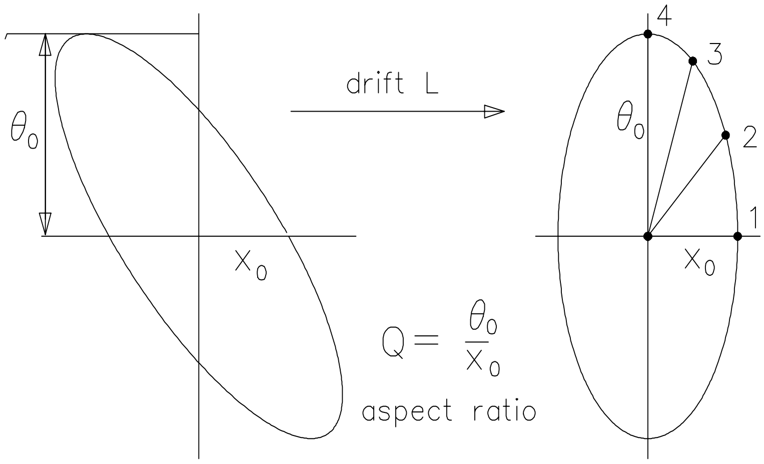


Figure 1: The phase ellipse description of a beam in the horizontal plane.

case the equation for the ellipse is:

$$x^2 + \frac{\theta^2}{Q^2} = x_0^2 \tag{1}$$

where Q is called the aspect ratio of the beam, given by

$$Q = \frac{\theta_0}{x_0} \tag{2}$$

The reduced emittance is:

$$\frac{\epsilon}{\pi} = x_0\theta_0 = Qx_0^2 = Q\left(x^2 + \frac{\theta^2}{Q^2}\right) \tag{3}$$

For a given value of Q, it is possible to associate with every point (x, θ) an upright ellipse which is congruent with the ellipse of equation 1 , determine a new x_0^2, and associate with that point a proportionally larger emittance. As will be shown in section 5 , this emittance fully determines the value of the angle α in the solenoid, when a particle described by this phase space point is injected into the solenoid.

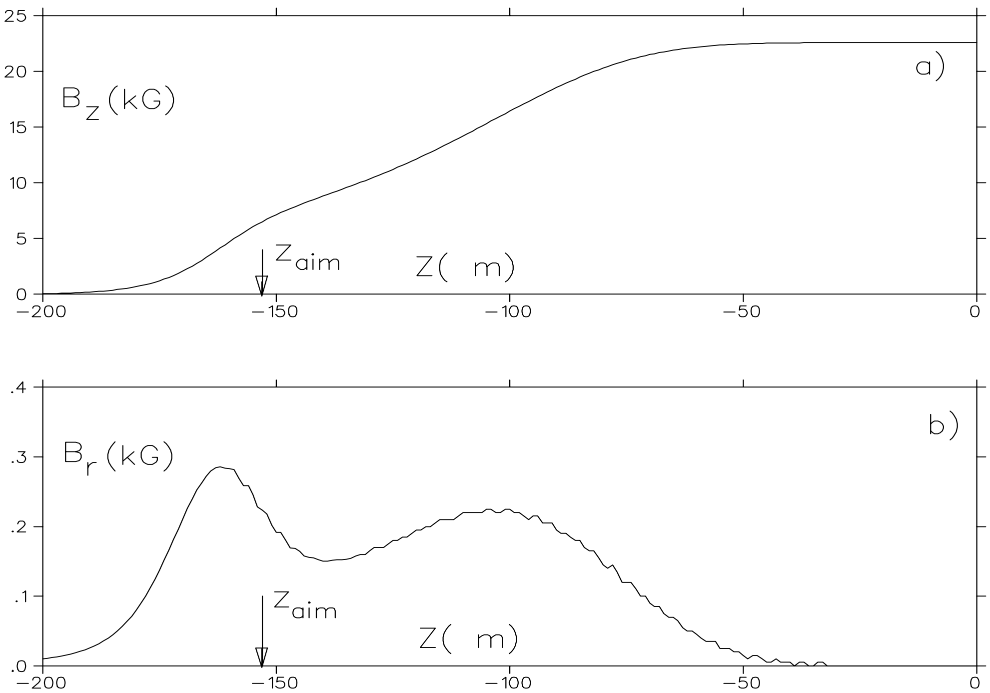


Figure 2: The longitudinal field B_{z} on the axis, and the radial field B_{r} at 2 cm from the axis

3 The magnetic field in the solenoid

The solenoid has a very large bore, and is surrounded by an iron joke with a 10 to 20 cm radius entrance hole about 1.5 m from the longitudinal center of the solenoid. Figure 2a shows the longitudinal field component on the axis, B_z, as function of z. The muons used in experiment E614 are close to the axis. For a small radius the radial field component B_r is:

$$B_r = \frac{r}{2}\frac{dB(z)}{dz} \tag{4}$$

B_r, which is shown in figure 2b, has not one but two peaks due to the distortion of the B_z field by the joke.

4 Optimization of beam phase ellipse for a given fringe field

The criterium for an optimal injection is a minimum angle α in the homogeneous part of the solenoid field. There are two aspects to this optimum. Firstly, the solenoid has to be positioned optimally with respect to the beam. The beam is focussed into the fringe field, and will come to a waist at some longitudinal position, when the solenoid field is off. Of course, when the field is on, the beam will never reach that position in that way. It is some kind of virtual waist. This is just an other way of saying that the phase ellipse at a well defined upstream position must have an optimal tilt angle. As already explained in section 2, the tilt angle determines the drift to the waist. The position of the waist has to be related to the longitudinal distance to the center of the solenoid. It is easy to establish the optimal position by ray-tracing. Take ray 4 of the phase ellipse of figure 1. It represents a ray in the horizontal plane which is crossing the axis. The angle α for this ray depends on the z-position, where it crosses the axis. This is shown in figure 3 , where α is plotted for some angle θ_0 as function of the longitudinal distance d in cm to the optimum position. The points can be fitted with a parabola. The point $d = 0$ corresponds to the point $z = -153$ upstream of the solenoid center and is indicated in figure 2 as z_{aim}.

Secondly, for a beam with a given emittance, all points on the circumference of the phase ellipse have to give the same α. In particular, ray 1 with $\theta = 0$, which represents a ray entering parallel to the axis has to give the same α as ray 4. This requirement gives a value x_0. The upright phase ellipse is now completely determined. Figure 4 shows that the 4 rays all end up with the same α. The aspect ratio Q is now the aspect ratio of the optimally shaped phase ellipse. An optimal beam with a larger emittance has to have the same aspect ratio. Therefore, both x_0 and θ_0 have then to be increased by the same amount. Q and z_{aim} are characteristic for a given fringe field distribution. For an other fringe field they will be different. Both depend on the momentum of the beam.

The vertical phase space of the beam has to be shaped in the same way as the horizontal phase space.

5 Equations for α, r, x and y

- Below, all angles are in mr, and all positions are in mm.

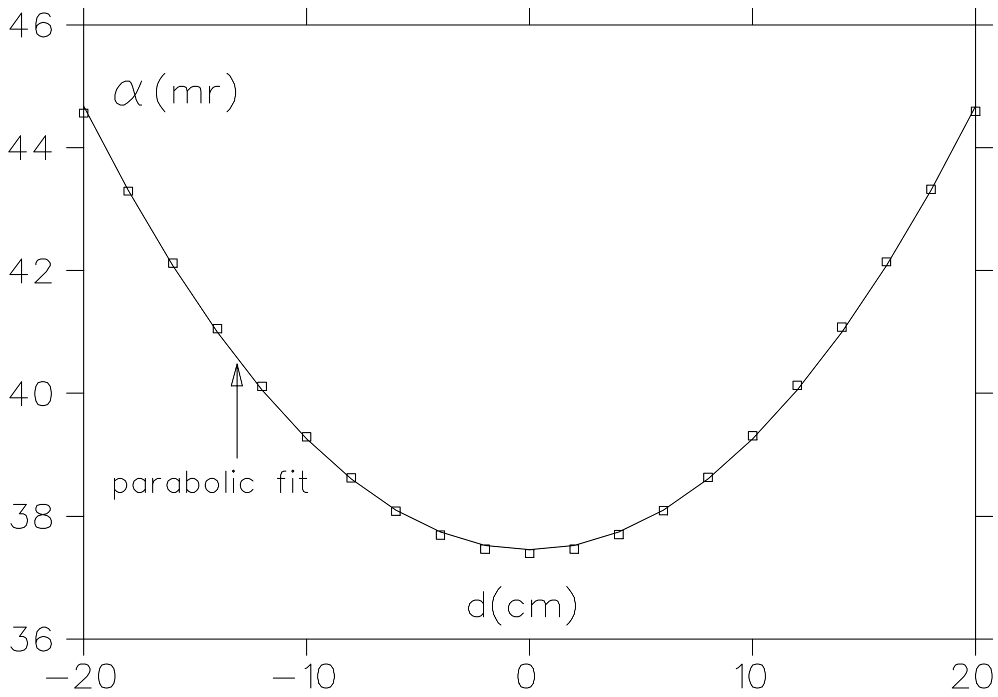
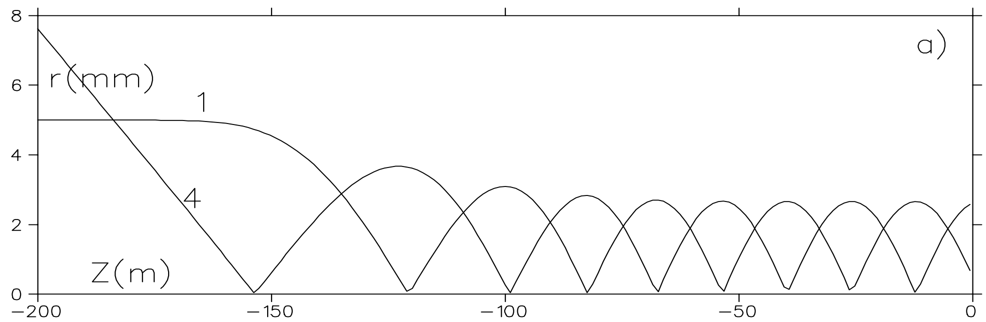


Figure 3: The angle α for ray 4 of 1 as function of the distance d to the optimum point z_{aim}

- The equations below were found empirically by raytracing particles into the solenoid with the raytracing program ZGOUBI, and comparing initial and final values of the relevant parameters.
- The equations below are for a zero percent momentum bite. For a finite momentum bite, a linear dependence of ρ and z_{ref} (defined below) on the momentum will give a drastic improvement.

After a particle passes through the fringe field into the area of homogeneous field, it spirals with a fixed angle α around a circle with a fixed radius r_{spiral}, centered at some point in the $x-y$ plane. In doing so, it has a radius r to the solenoid axis, and positions x_{f} and y_{f}. The position of the center is, of course, fixed. This is illustrated in figure 5



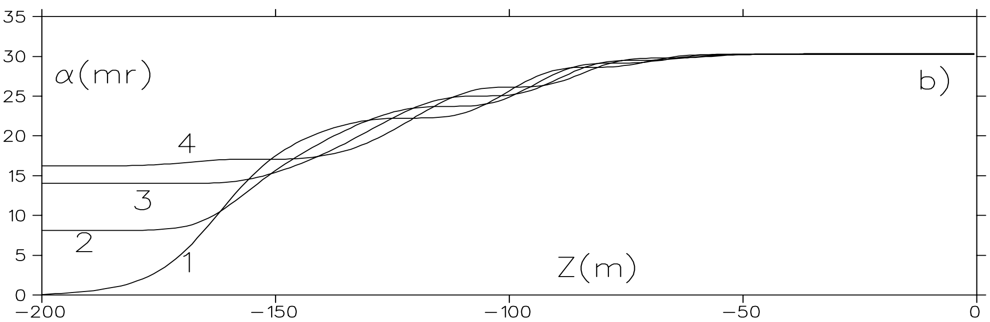


Figure 4: The angle α and the radius r for 4 rays on the circumference of the optimum phase ellipse

It is possible to express α, r, $x_{\rm f}$ and $y_{\rm f}$ as functions of the initial coordinates before the solenoid. Here, r is the distance to the solenoid axis. The expressions become simple if the particle is first traced to $z_{\rm aim}$ with the solenoid field off. In general, the particle will have now new coordinates x, y, θ and ϕ. A further simplification is possible because of the cylindrical symmetry by transforming the coordinates by a rotation in the $x-y$ plane to a coordinate system, in which $y=0$. The expressions will now only depend on x, θ and ϕ. As will be clear below, the expressions contain three reference parameters which are fixed for a given fringe field, namely Q, the aspect ratio of the optimum shape of the phase ellipse, $z_{\rm ref}$, and $\gamma_{\rm ref}$.

The expression for the angle α contains only Q and ρ.

Here, ρ is the radius of curvature of a particle with momentum P that moves in a plane perpendicular to the direction of the constant homogeneous

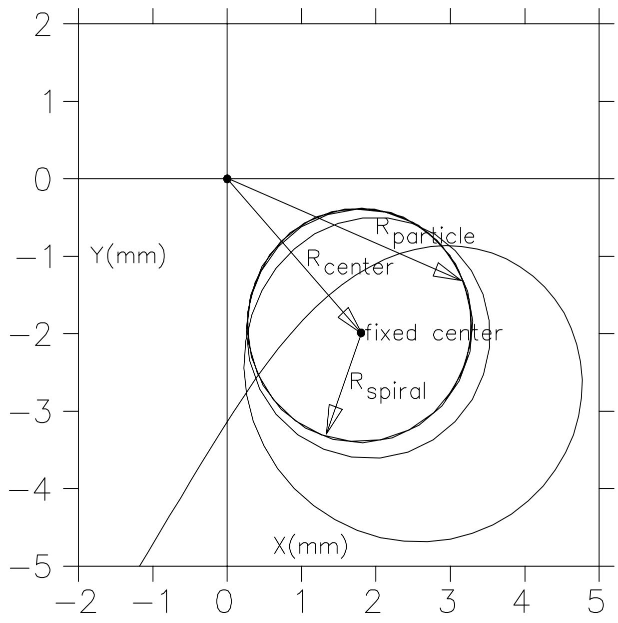


Figure 5: The $x-y$ projection of a particle spiraling in the solenoid

field B. It is given by

$$\rho(\mathrm{mm}) = 33.35641 \frac{P(\mathrm{MeV/c})}{B(\mathrm{kG})} \tag{5}$$

The equation for α is

$$\alpha = \sqrt{\frac{1000}{2\rho Q}\left[\theta^2 + (\phi - Qx)^2\right]} \tag{6}$$

In the case that $\phi = 0$, this reduces, using equation 3 , to:

$$\alpha = \sqrt{\frac{1000}{2\rho}\frac{\epsilon}{\pi}} \tag{7}$$

The parameter Q is not longer present. This means that the value of α is determined completely by the value of the emittance associated with the phase space point (x, θ). In particular, if a beam with a certain emittance is injected into two solenoids with the same central field but different fringe fields, the distribution for α in the central region will be exactly the same. This result was first found in Monte Carlo simulations for several very different fringe fields for a properly constructed 4-dimensional phase space, and is now explained by this formula.

The particle spirals around with a radius:

$$r_{\mathrm{spiral}} = 0.001\rho\alpha \tag{8}$$

The equation for the radius r with respect to the solenoid axis is:

$$r = \sqrt{\frac{2\rho Q}{1000}\left[\left(x^2 + \frac{\theta^2}{Q^2}\right)\sin^2(\chi+\psi) + \frac{\phi^2}{Q^2}\sin^2\psi\right]} \tag{9}$$

where

$$\chi = \arctan\left(\frac{Qx}{\theta}\right) \tag{10}$$

and

$$\psi = \frac{z + z_{\mathrm{ref}}}{2\rho} \tag{11}$$

The parameter ψ takes into account the movement of the particle around its spiral, as it progresses longitudinally. Here, χ is not a free parameter, but is completely determined by x and θ. $z = 0$ in the center of the solenoid. When $z = -z_{\mathrm{ref}}$, then $\psi = 0$. This means that r does not depend on the initial ϕ. Then r_{b}, which is defined below, is zero. In other words, r depends only on x and θ for $z = -z_{\mathrm{ref}}$. When, furthermore, $x = 0$, $\chi = 0$, and it follows that $r = 0$. This is the way that z_{ref} was determined by raytracing. For $\psi = 0$, r becomes

$$r = r_{\mathrm{a}} = \sqrt{\frac{2\rho}{1000}\frac{\epsilon}{\pi}}\sin(\chi) \tag{12}$$

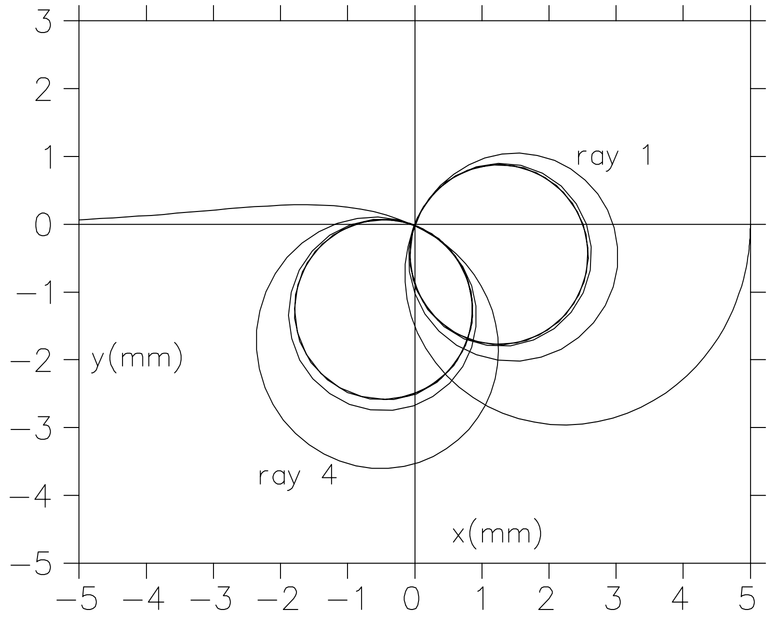


Figure 6: The $x - y$ projection of rays 1 and 4 in the solenoid. Ray 1 has $\theta = 0$. Ray 4 has $x = 0$

where r_{a} is given below.

If $\phi = 0$, there will always be a position z for which $r = 0$, since for a complete spiral $\sin(\chi + \psi)$ will always go though zero. In other words, if the original movement is completely in the horizontal plane, then the ray will always go through the axis. This is demonstrated in figure 6 for rays 1 and 4 of figure 1. The two circles are rotated over 90 degrees with respect to each other.

However, if ϕ is not zero, the ray will never go through the axis, except for the trivial case that $x = 0$.

Figure 4 shows that the radius for ray 1 with $\theta = 0$, giving $\chi = 90$ degrees, is 90 degrees out of phase with the radius for ray 2 with $x = 0$, giving $\chi = 0$ degrees. This is in agreement with equation 12.

ϕ enters in the formula in quadrature. This means that r is the same for $+\phi$ and $-\phi$. However, according to equations 6 and 8 , the spiral radius is very different. Therefore, one particle moves along a big circle, and the

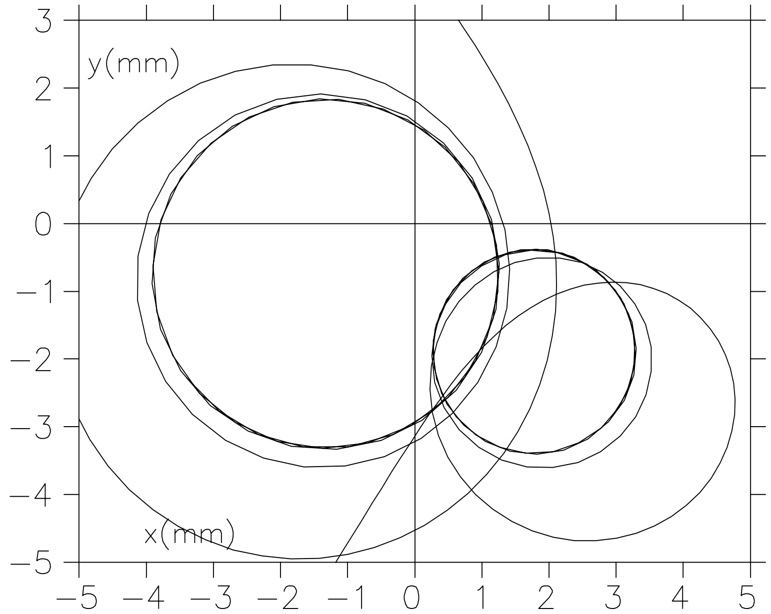


Figure 7: The $x - y$ projection in the solenoid of two rays with opposite ϕ but the same θ and x

other along a small circle. See figure 7. The origin lies inside the big circle, but outside the small circle.

The interesting fact is, that nevertheless at all times, i.e. for all z, the value of r is the same for both particles. A special case occurs when $\theta = 0$, but x and ϕ not zero. If $\phi = +Qx$, then $\alpha = 0$ and, therefore, the radius of the spiral is zero. The particle moves in a straight line parallel to the axis of the solenoid at a fixed point in the (x, y)-plane. Its distance to the axis is:

$$r_{\text{dis}} = x\sqrt{\frac{2\rho Q}{1000}} \tag{13}$$

The corresponding particle with $\phi = -Qx$ moves along a spiral with radius r_{dis} centered on the axis.

Equation 12 for r can be written in the form

$$r = \sqrt{r_{\mathrm{a}}^2 + r_{\mathrm{b}}^2} \tag{14}$$

where

$$r_{\mathrm{a}} = \sqrt{\frac{2\rho Q}{1000}\left(x^2 + \frac{\theta^2}{Q^2}\right)} \sin(\chi + \psi) \tag{15}$$

$$r_{\mathrm{b}} = \sqrt{\frac{2\rho Q}{1000}\left(\frac{\phi^2}{Q^2}\right)} \sin\psi \tag{16}$$

The coordinates x_{f} and y_{f} in the homogeneous part of the magnetic field can then be expressed as

$$x_{\mathrm{f}} = r_{\mathrm{a}} \cos\gamma - (\mathrm{sign}\phi) r_{\mathrm{b}} \sin\gamma \tag{17}$$

and

$$y_{\mathrm{f}} = r_{\mathrm{a}} \sin\gamma + (\mathrm{sign}\phi) r_{\mathrm{b}} \cos\gamma \tag{18}$$

where

$$\gamma = \gamma_{\mathrm{ref}} - \psi \tag{19}$$

The parameter γ takes into account the orientation of the center of the spiral in the $x - y$ plane.

If $\phi = 0$, it follows from equations 3 and 12 that, for a given z , the radius r varies sinusoidally with χ as the phase space point moves along the circumference of the phase ellipse. It follows from equations 17 and 18, that $y_{\mathrm{f}} = x_{\mathrm{f}} \tan\gamma$, which means that the point (x, y) moves along a straight line.

The above equations can further be used to predict other interesting behaviours.

6 The average depolarization of a beam in the the solenoid

The depolarization for a single particle is, using equation 6,

$$\text{depolarization} = 1 - \cos\alpha(\text{rad}) = \frac{\alpha^2}{2} = \frac{0.001}{4Q\rho}\left[\theta^2 + (\phi - Qx)\right] \qquad (20)$$

This gives for the average depolarization of an ensemble of particles

$$\text{average depolarization} = \frac{1}{N}\sum_N \frac{0.001}{4Q\rho}\left[\theta_N^2 + (\phi_N - Qx_N)\right] \qquad (21)$$

The uncertainty in the depolarization follows then from the uncertainty in the measured values of θ_{N}, ϕ_{N} and x_{N}.

7 Application of the equations to a beam from muon channel M13

The beam from M13 was traced with ZGOUBI from the production target to the center of the solenoid. A number of slits were closed to give an intensity of about 6,000 muons per second in a 1.0 percent momentum bite. In order to compare with the equations, the beam was alternatively traced to the exit of the last quadrupole, Q7. From the coordinates at Q7 the coordinates were calculated at the position z_{aim} in the solenoid. If y was not zero, the coordinate system was rotated to a system with $y = 0$. The formulas were applied to calculate angles and positions in the center of the solenoid. Finally, they were transformed back by rotation to the original coordinate system. Ideally, one should take into account the momentum dependence of the formulas, as discussed at the start of section 5. However, since it is not possible to measure the momentum at the exit of Q7, the formulas for a zero percent momentum bite were used. It turns out, that this still gives very good results, especially for the angle α.

Comparisons of the results for the two methods are shown in figure 8 and figure 9.

Figure 8a shows α versus r. Figure 8b shows dα versus dr, the differences for the two methods. These differences are mainly due to the finite momentum bite, as can be seen in figures 8c and 8d, where dα and dr are plotted against the momentum. Histograms of dα and dr are plotted in figure 9.

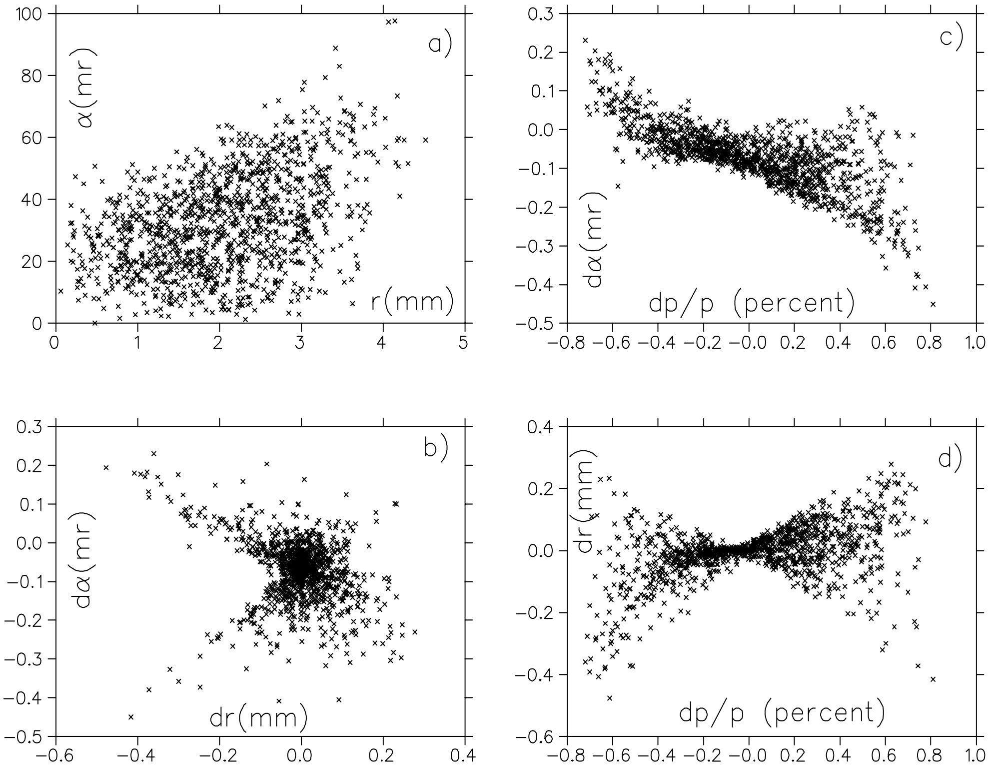


Figure 8: Comparison of α and r for the two methods.

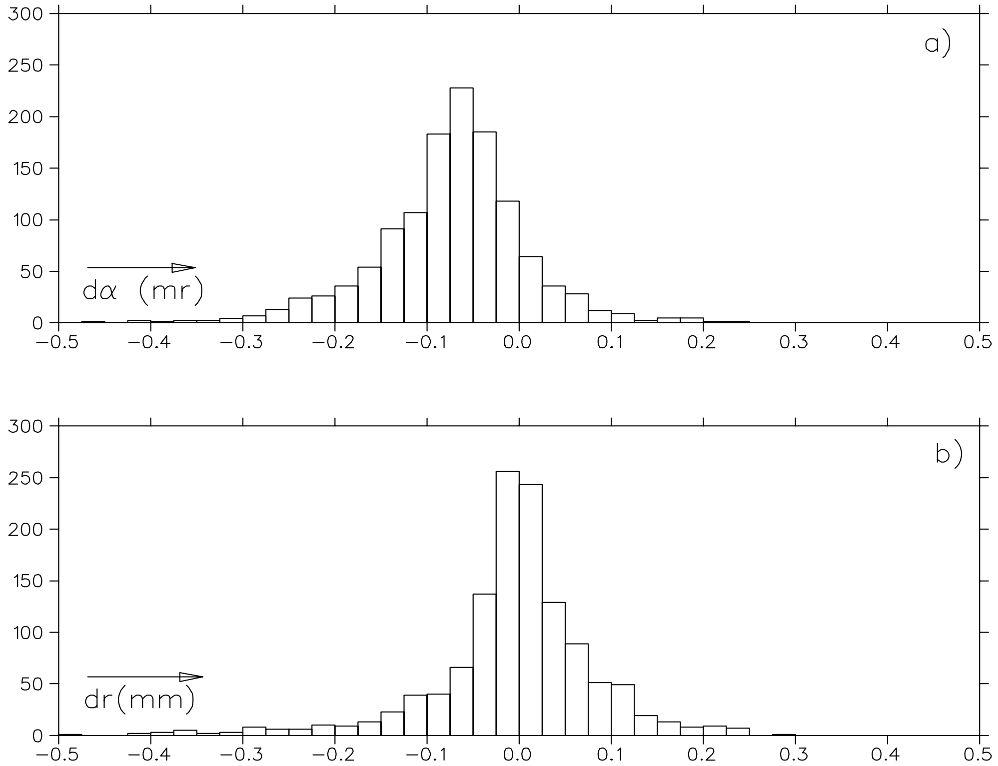


Figure 9: The differences in α and r for the two methods.