Radiative corrections in muon decay: a comparison

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1 The problem

Radiative corrections (due to the electromagnetic interaction) are important for the E614 experiment since they are responsible for significant modifications to the positron spectrum and asymmetry (see figures).

When I tried to compare various formulae I had some surprises. I was particularly concerned because the formulae used in the E614 Monte Carlo code were different from those found in published papers.

After more scrutiny I conclude, however, that the formulae are equivalent, except for the formula given by Källén[1]. There is a difference in one sign which may be explained by a misprint.

On the other hand I do not find any discrepancy between the formulae given by Scheck[2], Kalyniak and Ng[3] and the one used in the E614 Monte Carlo code[4].

2 The formula for the decay probability

1. Källén[1]. See page 92.

$$I(x) = \frac{g^2 m_\mu^5}{96\pi^3} \frac{d\Omega_e}{4\pi}$$

$$x^2 \left[3 - 2x + \frac{\alpha}{2\pi} f(x) + P \cos\Theta \left(1 - 2x + \frac{\alpha}{2\pi} g(x) \right) \right]$$

2. Scheck[2]. See page 211. Dropping the ϕ dependence:

$$\frac{d^2\Gamma(x,\Theta)}{dx d\cos\Theta} = \frac{G^2 m^5}{384\pi^3}$$

$$\left[x^2(3-2x) + f_c(x) \right] + \left[x^2(1-2x) + g_c(x) \right] \cos\Theta$$

3. Kalyniak and Ng[3] (for the μ^- decay). See page 1307:

$$\frac{d^2 R}{d\cos\Theta dx} = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

$$x^2 \left[3 - 2x + \frac{\alpha}{\pi} f_K(x) - \vec{\sigma}_\mu \cdot \hat{p}_e \left(1 - 2x + \frac{\alpha}{\pi} g_K(x) \right) \right]$$

4. E614 Monte Carlo[4] with $x_0 = 0$:

$$\begin{aligned} xF &= x^2(3-2x) + f_c(x) \\ xG &= x^2(2x-1) - f_\Theta(x) \end{aligned}$$

All these formulae can be put in the form (ignoring the constant factor for the moment):

$$S(x) = x^2 \left[3 - 2x + \frac{\alpha}{2\pi} f(x) + P \cos\Theta \left(1 - 2x + \frac{\alpha}{2\pi} g(x) \right) \right]$$

3 Formula for $f(x)$

1. Källén[1]:

$$f(x) = 2(3-2x)R(x)$$

$$-3\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x - 34x^2\right]$$

2. Scheck[2]:

$$f(x) = 2(3-2x)R(x)$$

$$-3\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

3. Kalyniak and Ng[3]:

$$f(x) = 2(3-2x)R(x)$$

$$-3\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

4. E614 Monte Carlo[4]:

$$f'(x) = 2(3-2x)R'(x)$$

$$+6(1-x)\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

4 Formula for $g(x)$

1. Källén[1]:

$$g(x) = 2(1-2x)R(x)$$

$$-\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

2. Scheck[2]:

$$g(x) = 2(1-2x)R(x)$$

$$-\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

3. Kalyniak and Ng[3]:

$$g(x) = 2(1-2x)R(x)$$

$$-\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

4. E614 Monte Carlo[4]:

$$g'(x) = 2(1-2x)R'(x)$$

$$+2(1-3x)\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

5 Formula for $R(x)$

1. Källén[1]:

$$R(x) = \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x} + \frac{3}{2}\right) + \log(1-x)\left(\log x + 1 - \frac{1}{x}\right) - \log x$$

$$-2\phi(-x) - \frac{\pi^2}{2} - \frac{1}{2}$$

2. Scheck[2]:

$$R(x) = \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x} + \frac{3}{2}\right) + \log(1-x)\left(\log x + 1 - \frac{1}{x}\right) - \log x$$

$$+2L_2(x) - \frac{\pi^2}{3} - \frac{1}{2}$$

3. Kalyniak and Ng[3]:

$$R(x) = \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x} + \frac{3}{2}\right) + \log(1-x)\left(\log x + 1 - \frac{1}{x}\right) - \log x$$

$$+2L_2(x) - \frac{\pi^2}{3} - \frac{1}{2}$$

4. E614 Monte Carlo[4]:

$$R'(x) = \log\left(\frac{m_\mu}{m_e}\right)\left(2\log\frac{1-x}{x} + \frac{3}{2}\right) + \left(3\log x - 1 - \frac{1}{x}\right)\log(1-x)$$

$$-\log x\left(2\log x - 1\right)$$

$$+2L_2(x) - \frac{\pi^2}{3} - 2$$

The formula for $R'(x)$ used in the E614 Monte Carlo[4] can be modified to:

$$R'(x) = \log\left(x\frac{m_\mu}{m_e}\right)\left(2\log\frac{1-x}{x}+\frac{3}{2}\right) - \log x\left(2\log(1-x) - 2\log x + \frac{3}{2}\right)$$

$$+\left(3\log x - 1 - \frac{1}{x}\right)\log(1-x) - \log x\left(2\log x - 1\right)$$

$$+2L_2(x) - \frac{\pi^2}{3} - 2$$

$$= \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x}+\frac{3}{2}\right) + 2\log(1-x) - 2\log x + \frac{3}{2}$$

$$-\log x\left(2\log(1-x) - 2\log x + \frac{3}{2}\right) + \left(3\log x - 1 - \frac{1}{x}\right)\log(1-x) - \log x\left(2\log x - 1\right)$$

$$+2L_2(x) - \frac{\pi^2}{3} - 2$$

$$= \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x}+\frac{3}{2}\right) + 2\log(1-x) - 2\log x + \frac{3}{2} - 2\log x\log(1-x)$$

$$+2\log^2 x - \frac{3}{2}\log x + \log(1-x)\left(3\log x - 1 - \frac{1}{x}\right) - 2\log^2 x + \log x$$

$$+2L_2(x) - \frac{\pi^2}{3} - 2$$

$$= \left[\log\left(x\frac{m_\mu}{m_e}\right) - 1\right]\left(2\log\frac{1-x}{x}+\frac{3}{2}\right) + \log(1-x)\left(\log x + 1 - \frac{1}{x}\right) - \frac{5}{2}\log x$$

$$+2L_2(x) - \frac{\pi^2}{3} - \frac{1}{2}$$

or:

$$R'(x) = R(x) - \frac{3}{2}\log x$$

If this expression is used in the expression of $f'(x)$ one has:

$$f'(x) = 2(3-2x)\left[R(x) - \frac{3}{2}\log x\right]$$

$$+6(1-x)\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

or:

$$f'(x) = 2(3-2x)R(x) - 3(3-2x)\log x$$

$$+6(1-x)\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

and finally:

$$f'(x) = 2(3-2x)R(x) - 3\log x$$

$$+\frac{1-x}{3x^2}\left[(5+17x-34x^2)\log\left(x\frac{m_\mu}{m_e}\right) - 22x + 34x^2\right]$$

Therefore:

$$f'(x) \equiv f(x)$$

Doing the same for the expression of $g'(x)$:

$$g'(x) = 2(1-2x)\left[R(x) - \frac{3}{2}\log x\right]$$

$$+2(1-3x)\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right) + 3 - 7x - 32x^2 + \frac{4}{x}(1-x)^2\log(1-x)\right]$$

or:

$$g'(x) = 2(1-2x)R(x) - 3(1-2x)\log x$$

$$+2(1-3x)\log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

and finally:

$$g'(x) = 2(1-2x)R(x) - \log x$$

$$-\frac{1-x}{3x^2}\left[(1+x+34x^2)\log\left(x\frac{m_\mu}{m_e}\right)+3-7x-32x^2+\frac{4}{x}(1-x)^2\log(1-x)\right]$$

Therefore:

$$g'(x) \equiv g(x)$$

6 Conclusion

There is an obvious discrepancy between the formula for $f(x)$ of Källén and those of Kalyniak-Ng and Scheck. On the other end there is agreement between Scheck and Kalyniak-Ng. There is probably a misprint in Källén's paper .

The formulae used in the E614 Monte Carlo code differ from those given by the other authors but the overall result is the same (expect for Källén due to the previously discussed discrepancy).

A more complete account on radiative corrections will be the subject of a technote.

7 Appendix: the Spence function

The Spence function ϕ is defined by:

$$\phi(z) = \int_1^z \frac{dt}{t}\log(1+t)$$

Another definition is:

$$L(z) = \int_0^z \frac{dt}{t}\log(1-t) = \phi(-z) + \frac{\pi^2}{12}$$

Kalyniak and Ng[3] use a different function:

$$L_2(z) = -\int_0^z \frac{dt}{t}\log(1-t) = -\phi(-z) - \frac{\pi^2}{12}$$

A comparison of the formulae gives:

$$2L_2(x) - \frac{\pi^2}{3} - \frac{1}{2} = -2\phi(-x) - \frac{\pi^2}{6} - \frac{\pi^2}{3} - \frac{1}{2} = -2\phi(-x) - \frac{\pi^2}{2} - \frac{1}{2}$$

So, the formulae are identical.

8 Figures

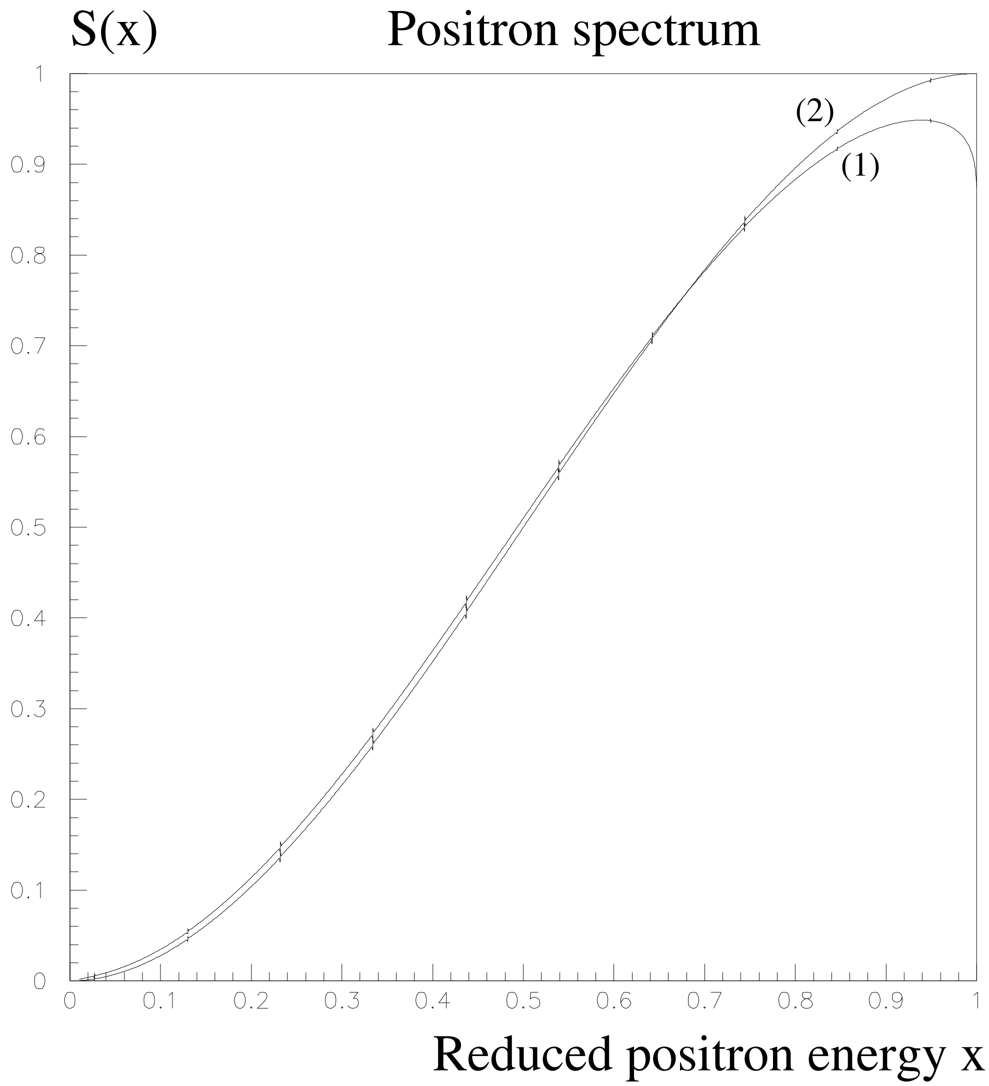

Figure 1: Positron spectrum with (1) and without (2) radiative corrections.

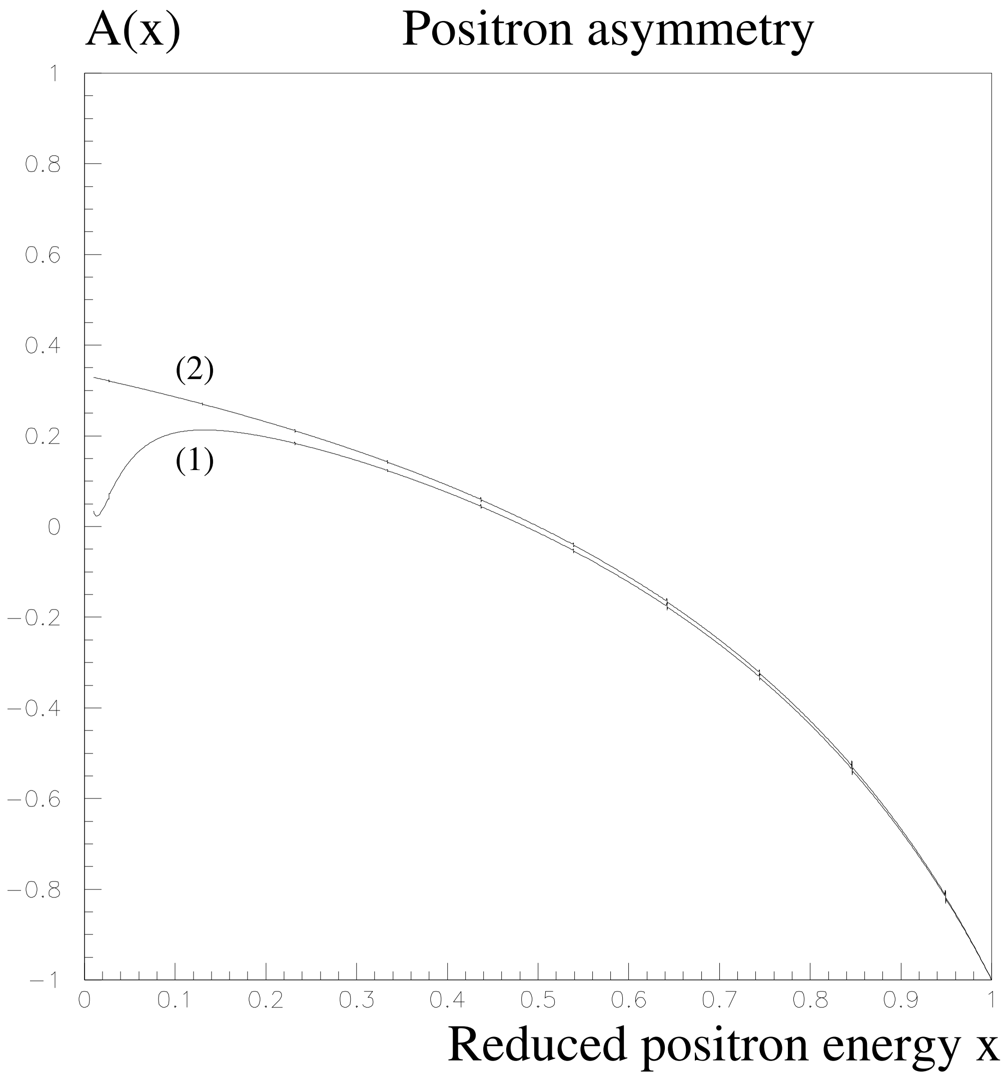

Figure 2: Positron asymmetry with (1) and without (2) radiative corrections.

References

[1] G. Källén, *"Radiative Corrections in Elementary Particle Physics"*, Springer Tracts in Modern Physics (Springer, Berlin, 1970), Vol. 46, p. 67.

[2] F. Scheck, *"Muon Physics"*, Phys. Rep. 44, 187, 1978.

[3] P. Kalyniak and J.N. Ng, *"Radiative corrections to muon decays with massive Dirac neutrinos"*, Phys. Rev. 25, 1305, 1982.

[4] TRIUMF-E614 Collaboration, Monte Carlo code, file gudcay.F.