

Consider two populations of longitudinally polarized muons at rest in the laboratory, one fully polarized along Oz ($N_1(t)$ at time t) and the other fully polarized in the opposite direction ($N_2(t)$ at time t). They both decay with the muon lifetime (decay rate λ_0), but, due to a spin-flip interaction, there is a transfer from one population to the other (transfer rate λ). This gives the coupled differential equations:

$$\begin{aligned}\frac{dN_1}{dt} &= -\lambda_0 N_1 - \lambda N_1 + \lambda N_2 \\ \frac{dN_2}{dt} &= -\lambda_0 N_2 - \lambda N_2 + \lambda N_1\end{aligned}\tag{1}$$

$$\begin{aligned}\frac{d}{dt}(N_1 + N_2) &= -\lambda_0(N_1 + N_2) \\ \frac{d}{dt}(N_1 - N_2) &= -(\lambda_0 + 2\lambda)(N_1 - N_2)\end{aligned}\tag{2}$$

The solution, including initial conditions, is:

$$\begin{aligned}N_1(t) + N_2(t) &= [N_1(0) + N_2(0)] \exp(-\lambda_0 t) \\ N_1(t) - N_2(t) &= [N_1(0) - N_2(0)] \exp[-(\lambda_0 + 2\lambda)t]\end{aligned}\tag{3}$$

and:

$$\frac{N_1(t) - N_2(t)}{N_1(t) + N_2(t)} = \frac{N_1(0) - N_2(0)}{N_1(0) + N_2(0)} \exp(-2\lambda t)\tag{4}$$

This is the degree of polarization $P(t)$ of the muon ensemble:

$$P(t) = P(0) \exp(-2\lambda t)\tag{5}$$

It decays with the decay constant 2λ .

Also:

$$\begin{aligned}N_1(t) &= \frac{1}{2} \exp(-\lambda_0 t) \left\{ N_1(0)[1 + \exp(-2\lambda t)] + N_2(0)[1 - \exp(-2\lambda t)] \right\} \\ N_2(t) &= \frac{1}{2} \exp(-\lambda_0 t) \left\{ N_1(0)[1 - \exp(-2\lambda t)] + N_2(0)[1 + \exp(-2\lambda t)] \right\}\end{aligned}\tag{6}$$

Assuming that the initial state is a pure state of polarization along Oz, the initial conditions are:

$$N_1(0) = 1 \quad N_2(0) = 0\tag{7}$$

this gives:

$$\begin{aligned}N_1(t) &= \frac{1}{2} \exp(-\lambda_0 t) [1 + \exp(-2\lambda t)] \\ N_2(t) &= \frac{1}{2} \exp(-\lambda_0 t) [1 - \exp(-2\lambda t)]\end{aligned}\tag{8}$$

or, by using the dimensionless variable τ :

$$\begin{aligned} N_1(t) &= \frac{1}{2} \exp(-\tau) [1 + \exp(-2\alpha\tau)] \\ N_2(t) &= \frac{1}{2} \exp(-\tau) [1 - \exp(-2\alpha\tau)] \end{aligned} \tag{9}$$

with $\lambda_0 t = \tau$ and $\alpha = \lambda/\lambda_0$.

In the following figures are plotted, for three different values of α :

1. $N_1(t)$: first population of muons.
2. $N_2(t)$: second population of muons.
3. $N_1(t) + N_2(t)$: total population of muons. It decays with the muon decay constant λ_0 .
4. $P(t) = (N_1(t) - N_2(t))/(N_1(t) + N_2(t))$: degree of polarization. It decays with the decay constant 2λ .

The time τ is in units of the muon lifetime.

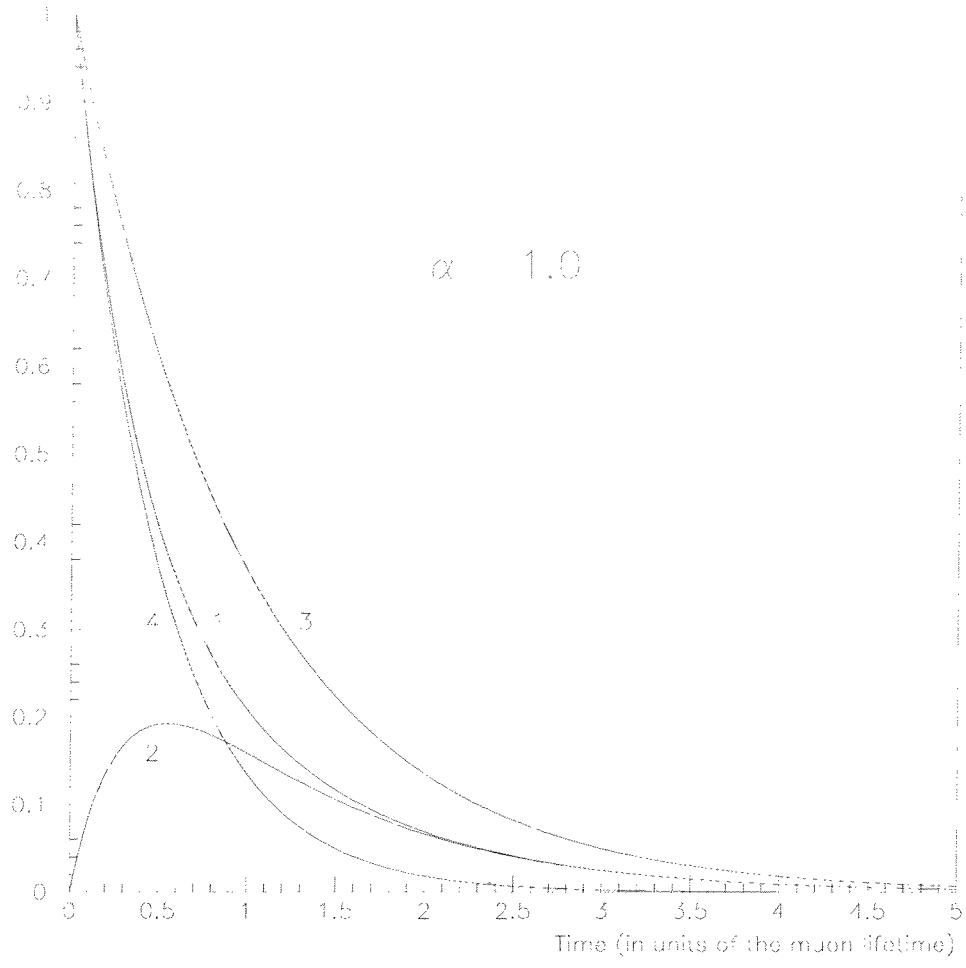


Figure 1: Time evolution of the two populations of muons. Case $\alpha = 1.0$.

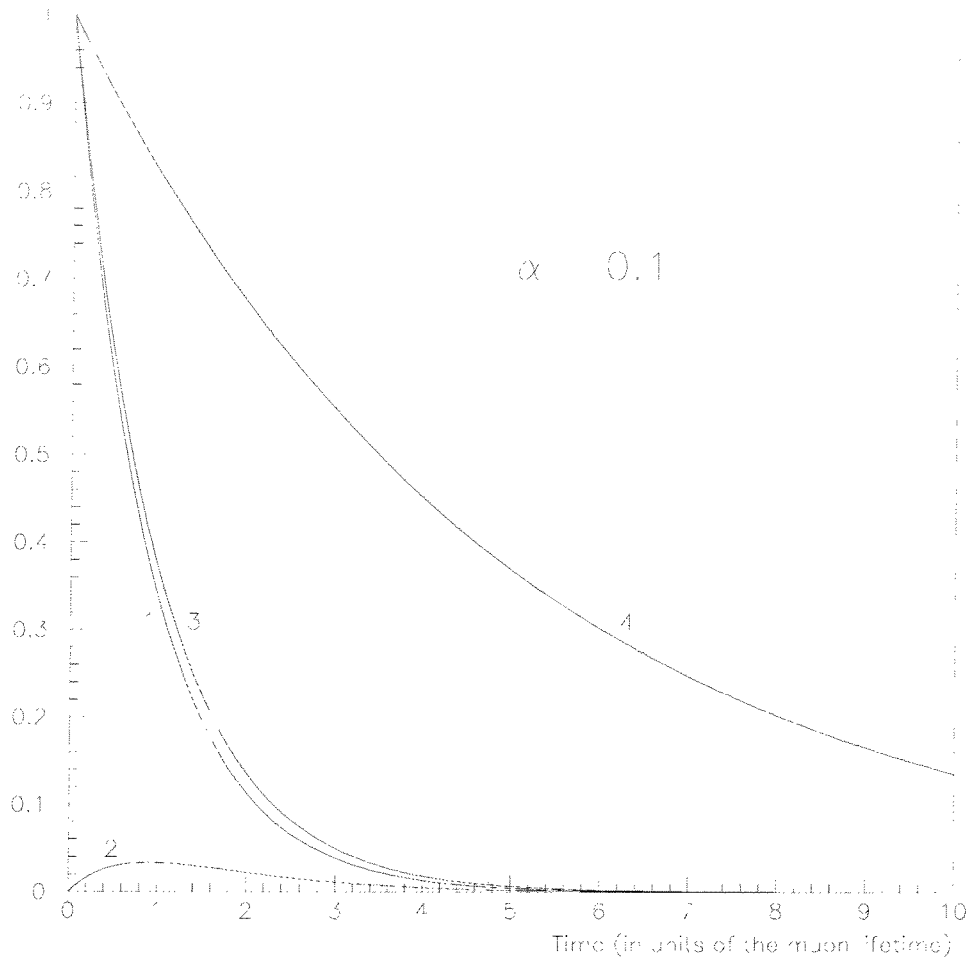


Figure 2: Time evolution of the two populations of muons. Case $\alpha = 0.1$.

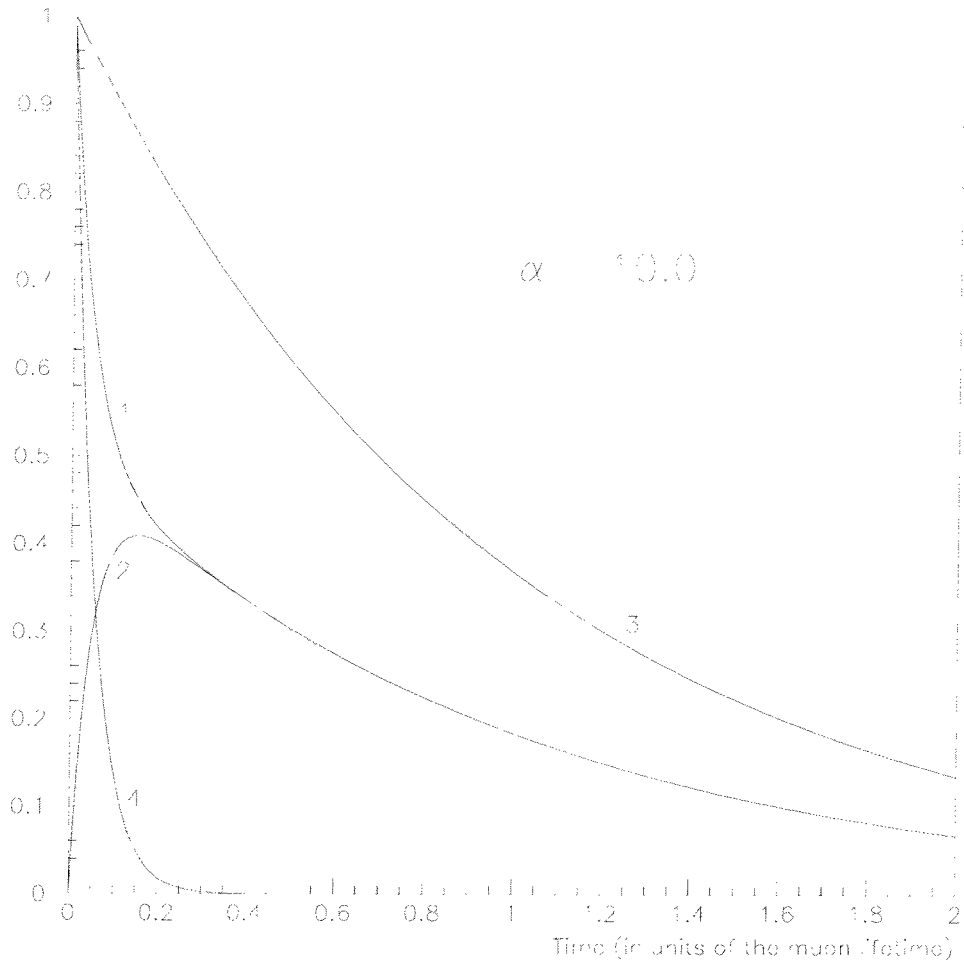


Figure 3: Time evolution of the two populations of muons. Case $\alpha = 10.0$.

Let us now study the shape of the positron spectrum and the positron asymmetry as a function of time. The general expression for the energy and angular distributions of the positron is:

$$\begin{aligned} \frac{d^2\Gamma}{dx d(\cos\theta)} = (x^2 - x_0^2)^{1/2} & \left\{ 6x(1-x) + \frac{4}{3} \rho(4x^2 - 3x - x_0^2) + 6\eta x_0(1-x) \right. \\ & \left. + P_\mu \xi \cos\theta (x^2 - x_0^2)^{1/2} \left[2(1-x) + \frac{4}{3} \delta \left(4x - 4 + (1 - x_0^2)^{1/2} \right) \right] \right\} \end{aligned} \quad (10)$$

In the following the positron mass will be neglected ($x_0 = 0$):

$$\frac{d^2\Gamma}{dx d(\cos\theta)} = x^2 \left\{ 6(1-x) + \frac{4}{3} \rho(4x - 3) + P_\mu \xi \cos\theta \left[2(1-x) + \frac{4}{3} \delta (4x - 3) \right] \right\} \quad (11)$$

Assuming the standard values for ρ and δ :

$$\frac{d^2\Gamma}{dx d(\cos\theta)} = x^2 \left[(3 - 2x) + P_\mu \xi \cos\theta (2x - 1) \right] \quad (12)$$

The asymmetry is:

$$A(x) = P_\mu \xi \cos\theta \frac{2x - 1}{3 - 2x} \quad (13)$$

For $P_\mu \xi = +1$:

$$\frac{d^2\Gamma}{dx d(\cos\theta)} = x^2 \left[(3 - 2x) + \cos\theta (2x - 1) \right] \quad (14)$$

$$A(x) = \cos\theta \frac{2x - 1}{3 - 2x} \quad (15)$$

For $\cos\theta = +1$:

$$\frac{d\Gamma}{dx} = 2x^2 \quad (16)$$

$$A(x) = \frac{2x - 1}{3 - 2x} \quad (17)$$

For $\cos\theta = -1$:

$$\frac{d\Gamma}{dx} = 4x^2(1 - x) \quad (18)$$

$$A(x) = -\frac{2x - 1}{3 - 2x} \quad (19)$$

The positron spectra for three values of $\cos \theta$ are shown on Figure 4.

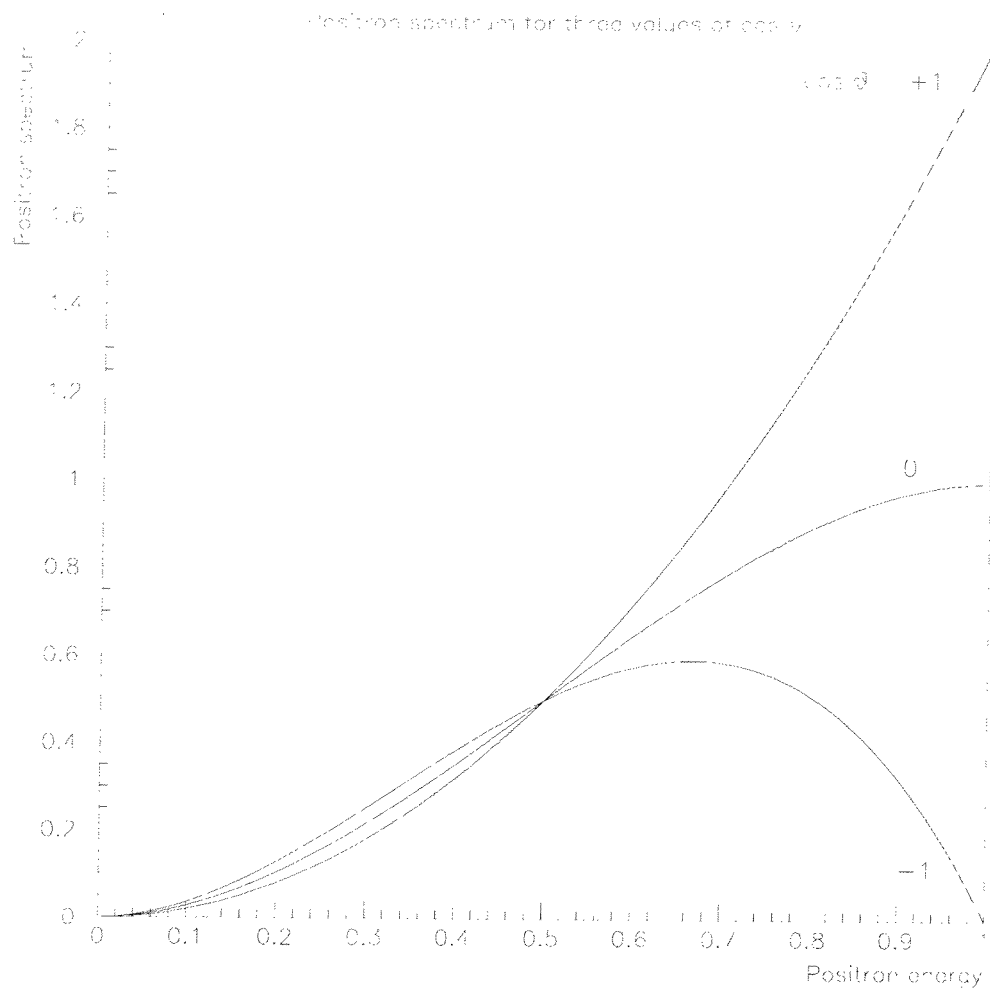


Figure 4: Positron spectrum for three values of $\cos \theta$.

The positron asymmetries for three values of $\cos \theta$ are shown on Figure 5.

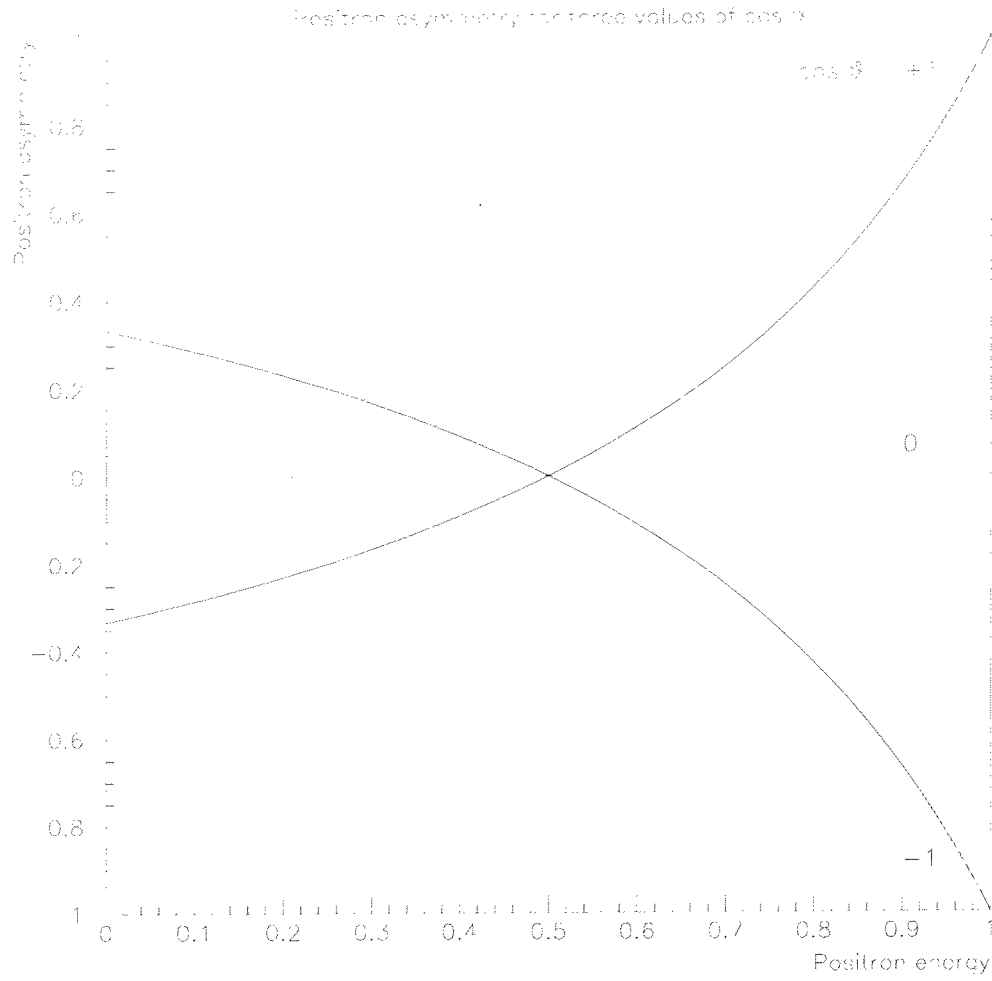


Figure 5: Positron asymmetry for three values of $\cos \theta$.

Assuming that positrons are detected in one direction, the positron spectrum will be a function of time:

$$S(x, t) = N_1(t)[2x^2] + N_2(t)[4x^2(1 - x)] \quad (20)$$

and the asymmetry:

$$A(x, t) = \frac{N_1(t) - N_2(t)}{N_1(t) + N_2(t)} \frac{2x - 1}{3 - 2x} \quad (21)$$

The following figures show the content of various energy bins (1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1) for two cases of initial conditions:

- $N_1 = 1, N_2 = 0.$
- $N_1 = 0, N_2 = 1.$

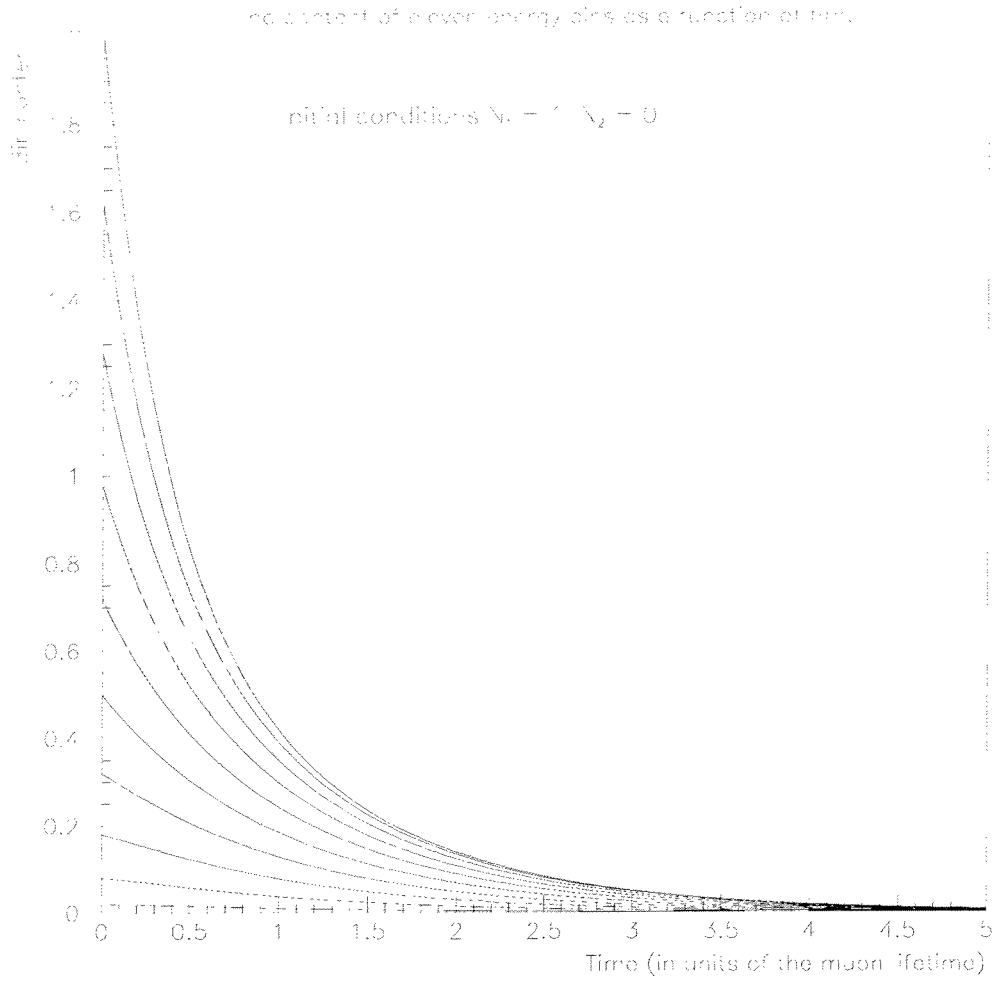


Figure 6: Content of eleven energy bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1, N_2 = 0$.

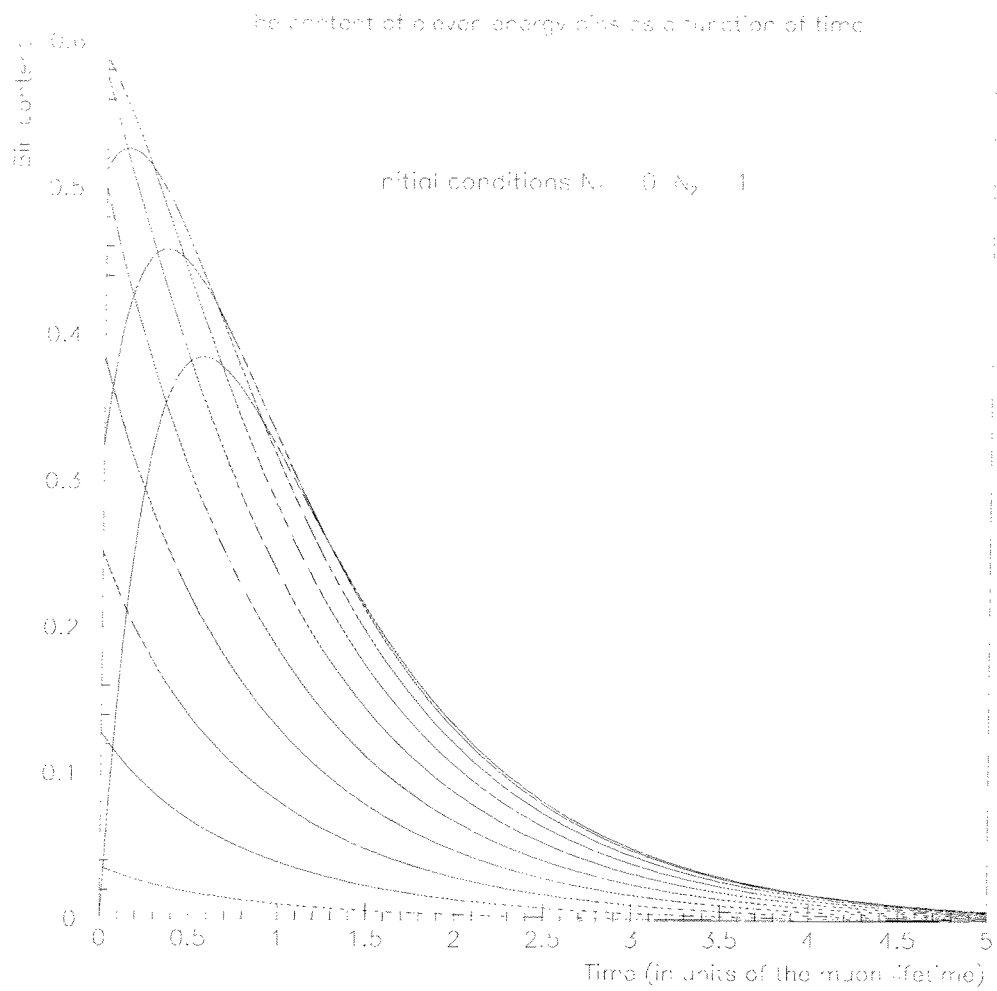


Figure 7: Content of eleven energy bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

The following figures show the content of various asymmetry bins (1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0,1) for two cases of initial conditions:

- $N_1 = 1, N_2 = 0.$
- $N_1 = 0, N_2 = 1.$

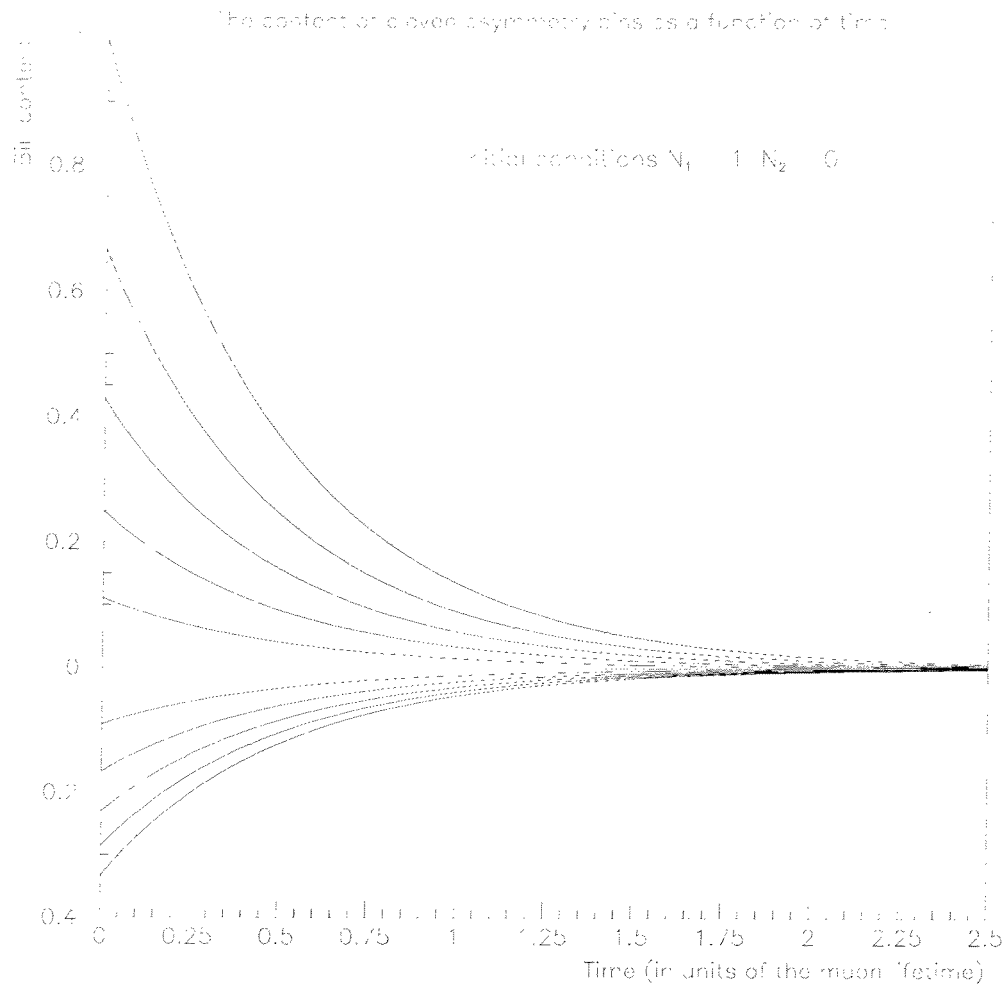


Figure 8: Content of eleven asymmetry bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

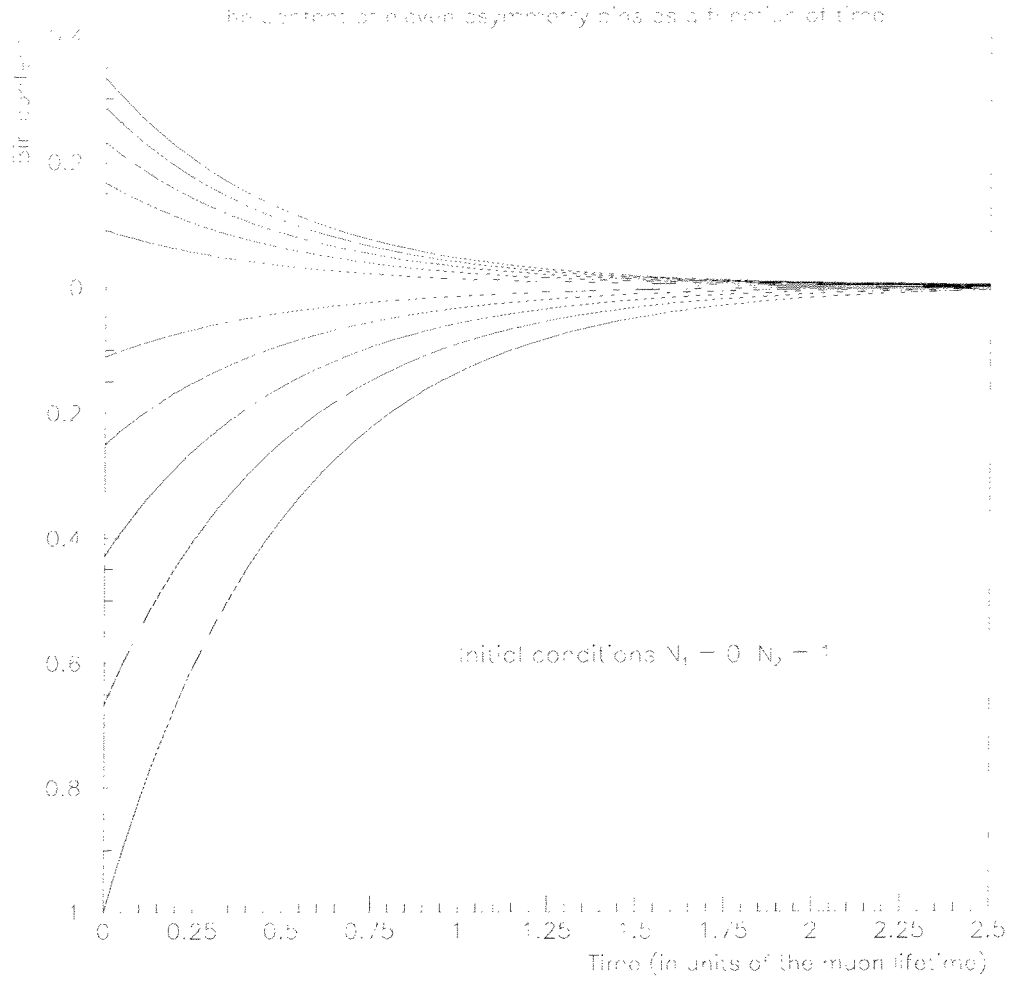


Figure 9: Content of eleven asymmetry bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

The following figures show the positron spectrum and positron asymmetry as functions of time for different initial conditions:

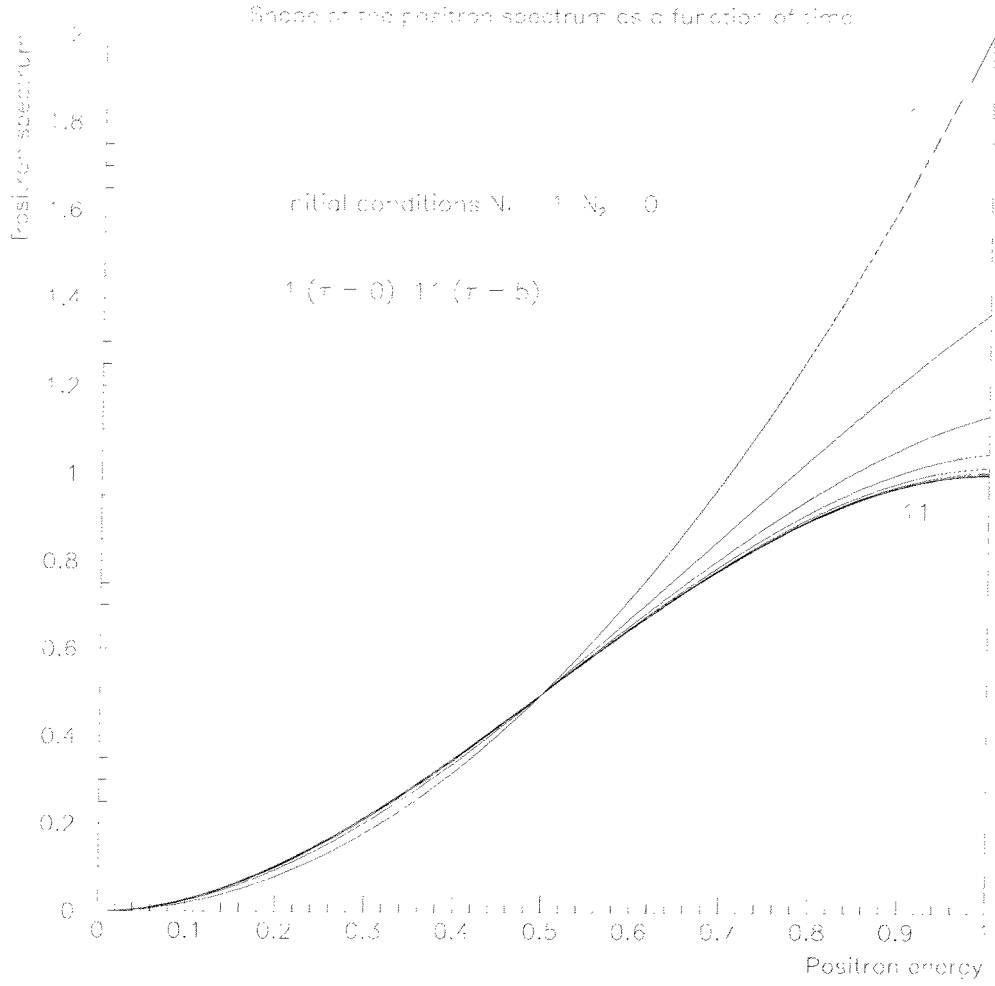


Figure 10: Positron spectrum. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

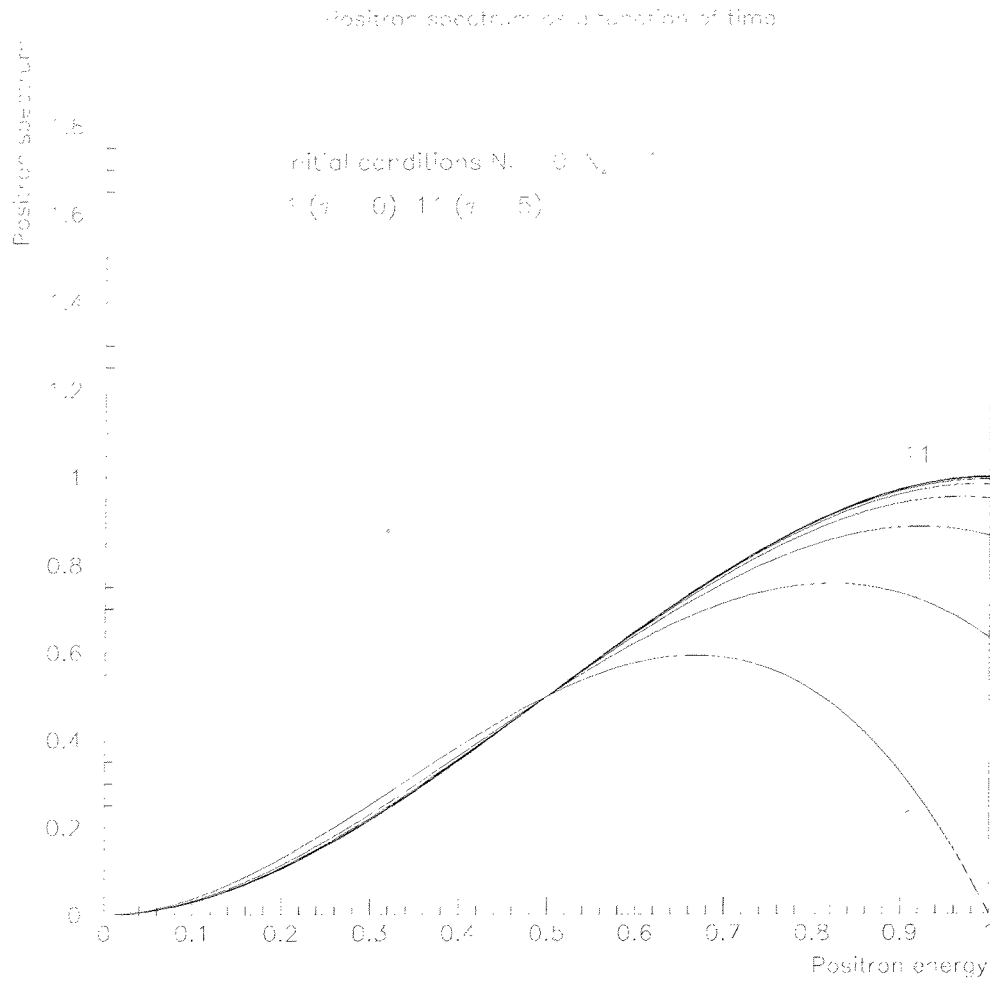


Figure 11: Positron spectrum. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0, N_2 = 1$.

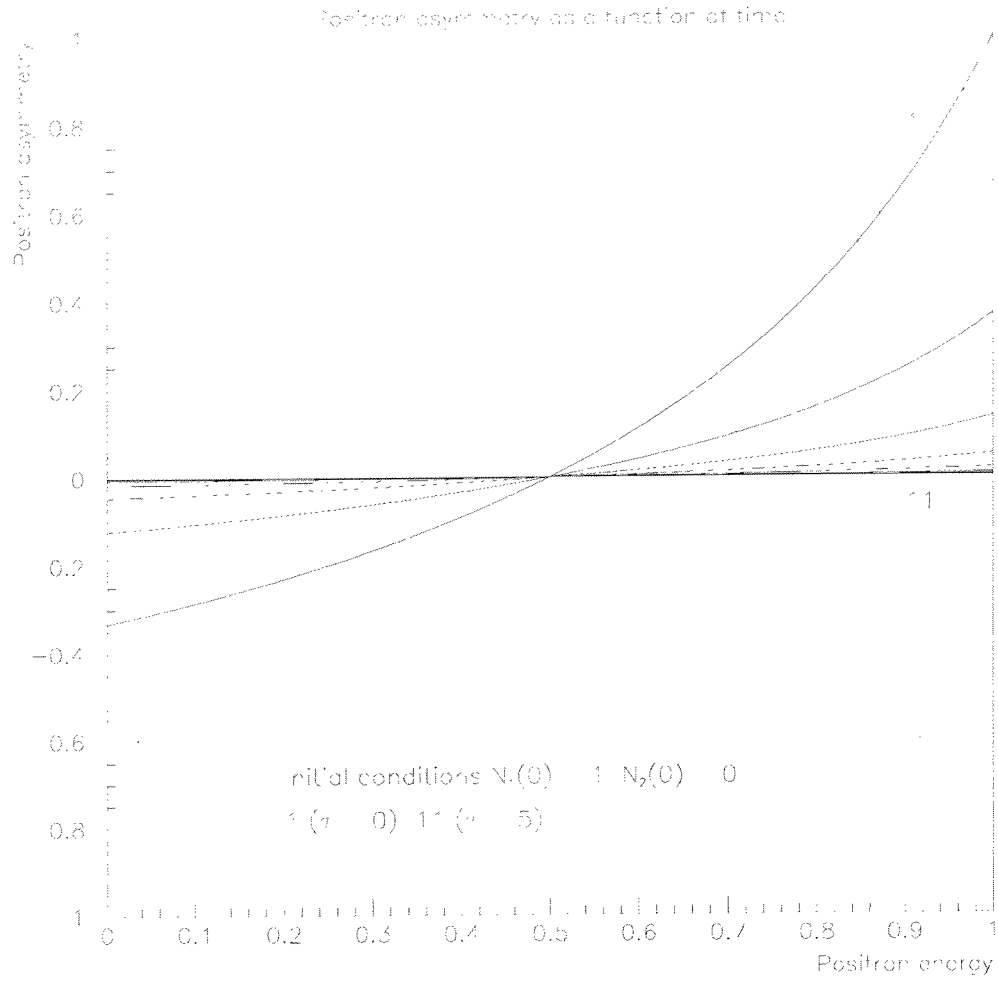


Figure 12: Positron asymmetry. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

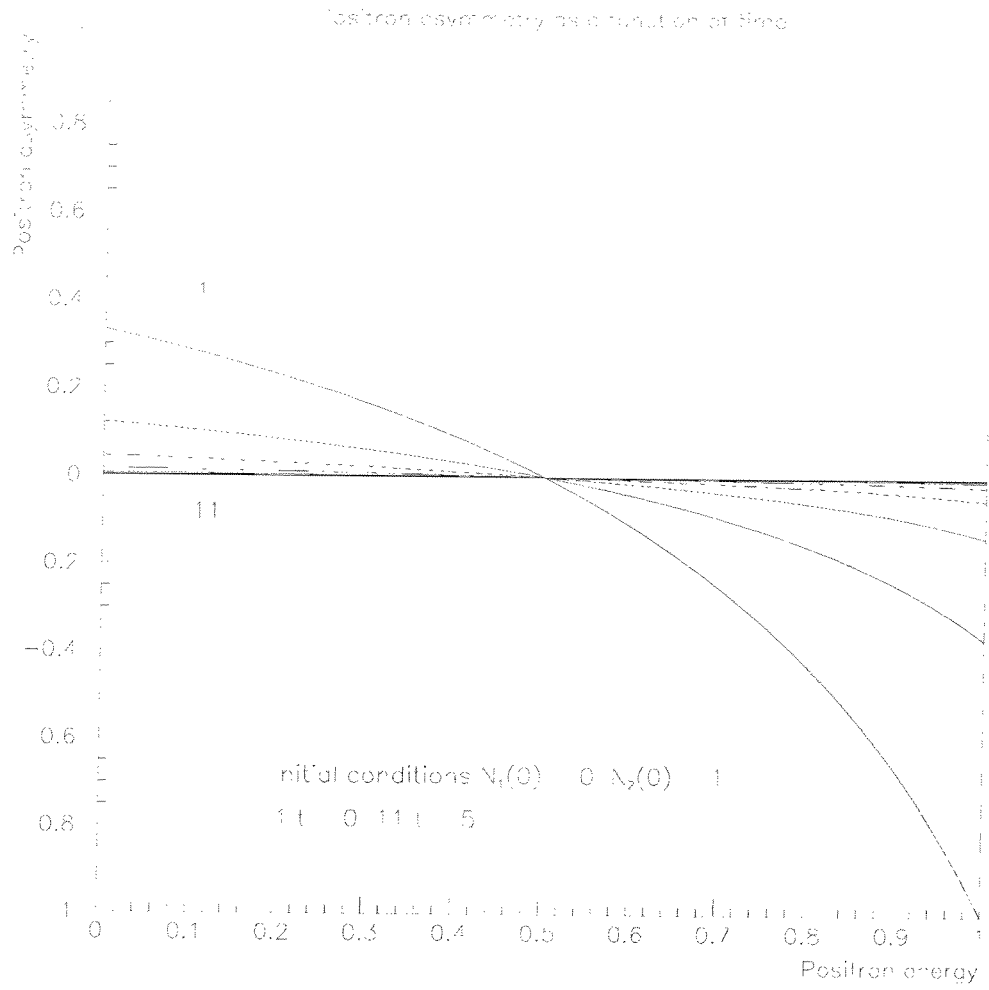


Figure 13: Positron asymmetry. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

November 29, 12:00