PRECESSION OF THE POLARIZATION OF PARTICLES MOVING IN A HOMOGENEOUS ELECTROMAGNETIC FIELD*

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The problem of the precession of the "spin" of a particle moving in a homogeneous electromagnetic field – a problem which has recently acquired considerable experimental interest – has already been investigated for spin $\frac{1}{2}$ particles in some particular cases.[1] In the literature the results were derived by explicit use of the Dirac equation, with the occasional inclusion of a Pauli term to account for an anomalous magnetic moment. On the other hand, following a remark of Bloch[2] in connection with the nonrelativistic case, the expectation value of the vector operator representing the "spin" will necessarily follow the same time dependence as one would obtain from a classical equation of motion. To solve the problem for arbitrary spin in the relativistic case, it will thus suffice to produce a consistent set of covariant classical equations of motion. Such equations have been indicated a long time ago by Frenkel[3] and are discussed by Kramers.[4] These authors use an antisymmetric tensor M as the relativistic generalization of the intrinsic angular momentum observed in the rest-frame of the particle. A formulation in terms of the (axial) four-vector s which describes the polarization in a covariant fashion[5] – though basically equivalent – is however much more convenient for our problem. We shall therefore derive first the equations of motion directly in terms of this four-vector s.

Let the spin of the particle be represented[6] in the rest-frame (R) by $\vec{s}$. We assume (a) that there exists a four-vector s such that in (R) it coincides with $\vec{s}$:

$$s=(s^0,\vec{s});\quad \text{in } (R),\ s=(0,\vec{s}). \tag{1}$$

Denoting the four-velocity of the particle by $u=(u^0,\vec{u})=\gamma(1,\vec{v})$ [where $\vec{v}$ is the ordinary velocity, and $\gamma(v)=(1-v^2)^{-1/2}$], one has in every frame

$$s\cdot u=0,\ \text{i.e.},\ s^0=\vec{s}\cdot\vec{v}. \tag{2}$$

We further assume (b) that $\vec{s}$ obeys in (R) the customary equation of motion

$$d\vec{s}/d\tau=(ge/2m)(\vec{s}\times\vec{H}), \qquad (R) \tag{3}$$

where $\vec{H}$, e, and m have their standard meanings, while the gyromagnetic ratio g is defined by this very equation. While s^0 vanishes by hypothesis in any instantaneous rest-frame, $ds^0/d\tau$ need not. In fact, (2) implies

$$ds^0/d\tau=\vec{s}\cdot(d\vec{v}/d\tau), \qquad (R) \tag{4}$$

for such frames. In general, $du/d\tau=f/m$ (where f = four-force), while in a homogenous external electromagnetic field specified by $F=-(\vec{E},\vec{H})$

$$du/d\tau=(e/m)F\cdot u. \tag{5}$$

The immediate generalization of Eqs. (3) and (4) to arbitrary frames is

$$ds/d\tau=(ge/2m)[F\cdot s+(s\cdot F\cdot u)u]-[(du/d\tau)\cdot s]u, \tag{6}$$

as can be checked by reducing to the rest-frame. With (5), one has for homogeneous fields

$$ds/d\tau=(e/m)[(g/2)F\cdot s+(g/2-1)(s\cdot F\cdot u)u]. \tag{7}$$

(5) and (7) constitute, for any value of g and arbitrary spin $\mathcal{S}$, a consistent set of equations of motion; they imply that $s\cdot s$ and $s\cdot u$ are constant, so that condition (2) is maintained.[7] For experiments of current interest, the main use of (7) is in the computation of the rate Ω at which longitudinal polarization is transformed into a transverse one (and vice versa). For this, we express s in the laboratory frame (L) in terms of two unit polarization four-vectors, e_l and e_t:

$$s/\mathcal{S}=e_l\cos\phi+e_t\sin\phi,$$

where

$$\mathcal{S}=(-s\cdot s)^{1/2},$$

$$e_l=\gamma(v,\vec{v}/v)\equiv\gamma(v,\hat{v}),\quad e_t=(0,\hat{n}),$$

$$\hat{n}\cdot\hat{n}=1,\quad \hat{n}\cdot\hat{v}=0. \qquad (L) \tag{8}$$

Clearly, $\Omega = d\phi/dt = d\phi/\gamma d\tau$. Introducing (8) into (7), and expressing all quantities as ordinary vectors, we find

$$\Omega = (e/m)\{(\vec{E}\cdot\hat{n}/v)[(g/2-1) - g/2\gamma^2] + (\hat{v}\cdot\vec{H}\times\hat{n})(g/2-1)\}. \quad (9)$$

The relevant "anomaly" of spin-$\frac{1}{2}$ particles, $(g/2-1)$, is clearly exhibited in (9) although our derivation was classical throughout.

We now specialize (9) to some cases of practical interest (the references are to experiments):

(A) $\vec{E}\times\vec{v} = \vec{H}\times\vec{v} = 0$; $\Omega = 0$. The character of the polarization does not change, but the transverse polarization precesses around $\vec{v}$ in longitudinal fields with an angular frequency $\omega = (ge/2m\gamma)H = (g/2)\omega_L$, as follows readily from (8).

(B)[8] $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\vec{E} = 0$; $\Omega = \omega_L(g/2-1)\gamma$, where ω_L is the Larmor frequency defined in (A) above.

(C)[9] $\vec{E}\cdot\hat{n} = E$, $\vec{H} = 0$; $\Omega = \omega_p[-g/2\gamma + (g/2-1)\gamma]$, where $\omega_p = eE/m\gamma v$ is the angular frequency of the particle's motion in the laboratory.

(D)[10] $\vec{E}\cdot\vec{H} = 0$, rectilinear motion: $\vec{E} = -\vec{v}\times\vec{H}$; $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\Omega = \omega_L(g/2\gamma)$.

(E)[11] $\vec{E}\cdot\vec{H} = 0$, $\vec{H}\cdot\hat{n}\times\hat{v} = H$, $\vec{E}\cdot\hat{n} = -Ev_x/v$, trochoidal motion: $E/H << 1$; $\Omega = (e/m)[(g/2-1)(H - Ev_x) + Ev_x/(\gamma^2-1)]$,

$$(\Delta\phi/2\pi) \text{ per loop} = \gamma(E/H)\gamma(v)[1-(E/H)v_x](g/2-1) = \gamma(v')(g/2-1),$$

where v' is velocity in a frame where $E' = 0$.

The generalization of (6) to cover particles having an intrinsic electric dipole moment $\vec{\epsilon} = (g'e/2m)\vec{s}$ may be of interest. In the (R) frame, the effect of $\vec{\epsilon}$ is taken into account by adding $\vec{\epsilon}\times\vec{E}$ to the right-hand side of (3), while leaving (4) unchanged. Thus the required change in the right-hand side of (6) is the addition of a term $-(g'e/2m)[(F^*\cdot s) + (s\cdot F^*\cdot u)u]$, denoting by F^* the dual of F, i.e., $F^* = -(\vec{H}, -\vec{E})$. For the experiment (B) above, one obtains then $|\Omega| = \omega_L\gamma \times[(g/2-1)^2 + (g'v/2)^2]^{1/2}$.

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[2]F. Bloch, Phys. Rev. 70, 460 (1946).

[3]J. Frenkel, Z. Physik 37, 243 (1926).

[4]H. A. Kramers, Quantum Mechanics (North Holland Publishing Company, Amsterdam, 1957), p. 226 et seq. Kramers' Eq. (4) (p. 229) does not correspond to our Eq. (6). His conclusion that already classically $g = 2$ is implied for an electron, based on the relativistic equation he uses, stems from the fact that that equation corresponds in the rest-frame to $ds^0/d\tau = (ge/2m)\vec{s}\cdot\vec{E}$. Comparing with our (4), with $d\vec{v}/d\tau = (e/m)\vec{E}$, one sees that the "derived" result is, in fact, built into the theory from the start. A more general equation is mentioned by Kramers on p. 231, in fine print, and attributed to Frenkel. The inconsistencies arising in Kramers' discussion of what he calls "spin-orbit" forces [i.e., of the form $(\nabla\vec{H})\cdot\vec{u}$ in the rest-frame] are connected with the fact that neither of his equations of motion applies when the field is inhomogeneous. In that case, $du/d\tau$ is not given by (5) alone, but has to include an additional term (the covariant analog of the gradient force just mentioned) before being introduced into (6).

[5]The covariant polarization four-vector s is essentially the expectation value of the operator w used by Bargmann and Wigner [Proc. Natl. Acad. Sci. U. S. 34, 211 (1948)] to characterize representations of the inhomogeneous Lorentz group: $\langle w\cdot w\rangle = -\mathcal{S}(\mathcal{S}+1)m^2$, $\mathcal{S}$ = spin. s can be expressed in terms of the skew tensor M of Frenkel (which satisfies $M\cdot u = 0$), and vice versa: $s = M^*\cdot u$, $M^* = s\times u$, i.e., $M^{*ik} = s^iu^k - s^ku^i$. For the quantum-mechanical applications of s see, e.g., C. Bouchiat and L. Michel, Phys. Rev. 106, 170 (1955).

[6]Our notation is: $c = 1$, $\hbar = 1$ throughout; coordinate four-vector of components $x^0 = t$, x^1, x^2, x^3: $x = (x^0, \vec{x})$, $\vec{x} = \{x^\alpha\}$ ($\alpha = 1, 2, 3$); metric of signature (+ ---); τ = proper time; a dot between symbols, contraction of neighboring indices with the metric tensor, e.g., $x\cdot x = (x^0)^2 - \vec{x}^2$; skew tensor of components T^{ik} indicated as $T = (\vec{T}^t, \vec{T}^s)$, $\vec{T}^t = \{T^{0\alpha}\}$, $\vec{T}^s = \{T^{\beta\gamma}\}$; $\alpha, \beta, \gamma = 1, 2, 3$; its dual by $T^* = (\vec{T}_s, -\vec{T}_t)$.

[7]Equations (5) and (7) can be integrated explicitly by reference to four orthonormal four-vectors $e^{(i)}$ such that each of them obeys (5), and $e^{(0)} = u$.

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