TWIST (E614) Experiment
Technical Note No.89

Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay
(Supplement to Technical Note No.82)

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1 Preamble

This technote is a supplement to Technote 82 [1]. Rather than updating Technote 82, it is more practical to present the additional information in a separate technote.

The question addressed in Technote 82 was: how would the radiative muon decay affect muon polarization, which, according to the Standard Model, is $P_L = -1$ for the normal decay $\pi^+ \to \mu^+\nu$ (the positive-muon polarization is longitudinal and reaches the maximum negative value -1)? It is expected that the contamination of the normal muon decay by the radiative decay would change the muon polarization. It can be concluded, however, that this effect is sufficiently small so that it can be neglected. This conclusion rests on the following arguments:

- The radiative muon decay has a small branching ratio. A recent measurement [2] gives:

$$\text{B.R.}(\pi^+ \to \mu^+\nu\gamma, E_\gamma > 1\text{MeV}) = \Big(2.0 \pm 0.24(\text{stat.}) \pm 0.08(\text{syst.})\Big) \times 10^{-4}$$

  The corresponding theoretical B.R.($E_\gamma > 1$ MeV) is 2.283×10^{-4}.

- The TWIST experiment uses a small part of the Dalitz plot (~ 1 per-cent).

- The longitudinal muon polarization P_L is close to -1 in a large fraction of the Dalitz plot, especially in that part of the Dalitz plot which is used in the experiment.

2 The branching ratio

The value which has been given in the litterature for a long time [3]:

$$\text{B.R.} = (1.24 \pm 0.25) \times 10^{-4}$$

is based on the emulsion experiment of Castagnoli and Muchnik [4]. This branching ratio corresponds to a cut of $< 435\mu$ on the muon track (a cut on the muon kinetic energy $T_\mu < 3.38$ MeV).

The most recent experimental value [2] is:

$$\text{B.R.}(E_\gamma > 1 \text{ MeV}) = \Big(2.0 \pm 0.24 \text{ (stat.)} \pm 0.08 \text{ (syst.)}\Big) \times 10^{-4} \quad (1)$$

to be compared with a theoretical prediction 2.283×10^{-4}, assuming a cut-off $E_\gamma < 1$ MeV.

This number is well reproduced by integrating the theoretical probability distribution (see Appendix 1 of Technote 82) based on the Internal Bremsstrahlung contribution only.

The value obtained by Castagnoli and Muchnik is also consistent with the theoretical prediction, assuming the above energy cut-off $T_\mu < 3.38$ MeV.

In contrast to the electronic decay mode $\pi^+ \to e^+\nu\gamma$, the muonic decay mode $\pi^+ \to \mu^+\nu\gamma$ is dominated by the Internal Bremsstrahlung (IB) contribution, and the Structure-Dependent (SD) contribution can be neglected (also the interference term between ID and SD can be neglected). The experimental result of Bressi et al. [2] is in agreement with this conclusion, not only in the branching ratio but also in the Dalitz plot distribution.

3 The kinematics

Let us recall the most important features of the kinematics.

An important parameter is the ratio:

$$r = \frac{m^2}{M^2} \quad (2)$$

where m is the lepton mass, M the meson mass. For the decay of interest:

m	(muon mass)	$= 105.66$ MeV
M	(pion mass)	$= 139.57$ MeV
r		$= 0.573$
$\sqrt{r}$		$= 0.757$

Be careful. In the paper of Bressi et al. [2], the parameter r has a different definition:

$$r = \frac{m}{M} \quad (3)$$

The kinematical (dimensionless) variables are defined as follows:

- Photon energy: $x = 2E_\gamma/M$, where E_γ is the true photon energy.
- Muon energy: $y = 2E_\mu/M$, where E_μ is the total energy of the muon (rest mass energy + kinetic energy).

The contour of the kinematically allowed region is given by:

$$y = (1-x) + \frac{r}{(1-x)} \tag{4}$$

and:

$$y = 1 + r \qquad x = \text{ any value} \tag{5}$$

or:

$$C(x,y) = (1-y+r)\Big[(1-x)(x+y-1) - r\Big] = 0 \tag{6}$$

On this contour, the momenta of the three particles are colinear, therefore the decay plane cannot be defined. The muon must be longitudinally polarized.

More details on the kinematics can be found in Technote 82 [1].

4 The differential decay rate

The relevant papers are those of Bardin and Bilen'kii [5], Bijnens et al. [6, 7] and Chen et al. [8]. In most cases the interest was in the kaon decays $K \to e(\mu)\nu\gamma$ but the formulae can also be used for the pion decays $\pi \to e(\mu)\nu\gamma$ after minor modifications: the ratio r has to be changed and the CKM matrix element V_{ud} must be used instead of V_{us}.

Different notations are used by different authors. For instance, Chen et al. use the variable:

$$\lambda = \frac{x+y-1-r}{x} \tag{7}$$

In the following we will consider the IB contribution only, for the reasons given above.

According to Bardin and Bilen'kii the double-differential decay rate is given by:

$$\frac{d^2\Gamma}{dxdy} = N(2r)\left(\frac{f_K}{m_K}\right)^2 IB(x,y) \tag{8}$$

with:

$$N = \left(\frac{\alpha}{32\pi^2}\right) G_F^2 |V_{ud}|^2 m_K^5 \tag{9}$$

The numerical value of N is calculated in Appendix 1 of [1].

The expression for $IB(x,y)$ is:

$$IB(x,y) = \frac{1-y+r}{x^2(x+y-1-r)}\left[x^2 + 2(1-x)(1-r) - \frac{2xr(1-r)}{x+y-1-r}\right] \tag{10}$$

Two important remarks:

- There is a singularity for $x = 0$ (x is the photon energy). This is characteristic of radiative processes. This singularity is compensated by a similar singularity which arises in the calculation of radiative corrections (internal photon loops) in the corresponding non-radiative process [9]. Therefore, in the context of this technote, this singularity is artificial.

- The rate vanishes for $y = 1+r$, that is where the muon reaches its maximal energy (it is also the energy of the muon in the non-radiative decay). In this particular configuration, the photon and the neutrino are emitted opposite to the muon momentum. But, in the IB process, the physics favors emission of soft photons at small angles with respect to the muon momentum. This is why this particular kinematics is suppressed.

If one neglects the lepton mass:

$$IB(x,y) = \frac{(1-y)(x^2-2x+2)}{x^2(x+y-1)} \tag{11}$$

This expression has been used for the $\pi \to e\nu\gamma$ decay.

Table 1 of [1] shows the population of the Dalitz plot.

5 The polarization vector of the muon

The longitudinal component of the polarization vector has been calculated by Bardin and Bilen'kii [5], and Chen et al., but not by Bijnens et al. Chen et al. [8] have calculated all three components, P_L (longitudinal), P_N (perpendicular to the momentum, in the decay plane), and P_T (transverse, perpendicular to the decay plane).

5.1 The longitudinal polarization

5.1.1 Bardin and Bilen'kii

Time-reversal violation is assumed (all form factors are real)

$$\vec{P}_L = P_L\left(\frac{\vec{v}}{v}\right) \tag{12}$$

where $\vec{v}$ is the velocity of the muon and P_L the degree of longitudinal polarization.

$$P_L = -N\left(\frac{d^2\Gamma}{dxdy}\right)^{-1}\frac{\sqrt{y^2-4r}}{y}(2r)\left(\frac{f_K}{m_K}\right)^2 PIB(x,y) \tag{13}$$

with the following expression for $PIB(x,y)$:

$$PIB(x,y) = \frac{1-y+r}{x^2(x+y-1-r)}\left[x^2+2(1+r-x)-\frac{2xyr}{x+y-1-r}\right. \tag{14}$$

$$\left.+2r\frac{xy-2(x+y-1-r)}{y^2-4r}\left(2-x-\frac{(1+r)x}{x+y-1-r}\right)\right] \tag{15}$$

Both terms $PIB(x,y)$ and $IB(x,y)$ vanish for $y = 1+r$ but the ratio has a finite value, and $P_L = -1$.

Table 2 of [1] shows the longitudinal polarization as a function of the photon energy and the muon kinetic energy.

5.1.2 Chen, Geng and Lih

The various components of the polarization vector are given by the ratios:

$$P_i(x,y) = \frac{\rho_i(x,y)}{\rho_0(x,y)} \qquad (i = L, N, T) \tag{16}$$

where L, N and T refer to the longitudinal, normal and transverse components, respectively.

$$\rho_0(x,y) = \frac{1}{2} e^2 G_F^2 \sin^2\theta_C (1-\lambda) \frac{4m_\mu^2 |f_K|^2}{\lambda x^2} \rho_{0IB}(x,\lambda) \tag{17}$$

with:

$$\rho_{0IB}(x,\lambda) = x^2 + 2(1-r)\left(1 - x - \frac{r}{\lambda}\right) \tag{18}$$

$$\rho_L(x,y) = -\frac{1}{2} e^2 G_F^2 \sin^2\theta_C \frac{(1-\lambda)}{\lambda\sqrt{y^2-4r}} \frac{4m_\mu^2 |f_K|^2}{\lambda x^2} \rho_{LIB}(x,\lambda) \tag{19}$$

with:

$$\rho_{LIB}(x,\lambda) = x(\lambda y - 2r)(x + y - 2\lambda) - (y^2 - 4r)(\lambda x + 2r - 2\lambda) \tag{20}$$

For the pion radiative decay $\sin^2\theta_C$ must be replaced by $\cos^2\theta_C$.

As far as P_L is concerned there is perfect agreement between the two calculations (Bardin and Bilen'kii, Chen et al.)

6 Acknowledgments

The author wishes to than John Ng for enlightening comments.

References

[1] *"Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay"*, TRIUMF-TWIST Technical Note No.82, P. Depommier, June 23, 2003.

[2] *"New measurement of the $\pi \to \mu\nu\gamma$ decay"*, G. Bressi et al., Nucl. Phys. B513, 555 (1998).

[3] *"Review of Particle Physics"*, Particle Data Group, Phys. Rev. D66, 010001 (2002), see page 010001-441.

[4] *"Experimental Results on the Radiative $\pi - \mu$ Decay"*, C. Castagnoli and M. Muchnik, Phys. Rev. 112, 1779 (1958).

[5] *"The Radiative Decays $\pi(K) \to l\nu_l\gamma$"*, D.Yu. Bardin and S.M. Bilen'kii, Sov. J. Nucl. Phys. 16, 311 (1973).

[6] *"Semileptonic Kaon Decays in Chiral Perturbation Theory"*, J. Bijnens et al., LNF-92/047 (P), 5 Giugno 1992 (Laboratori Nazionali di Frascati); CERN-TH.6504/92 (CERN); BUTP-92/20 (University of Bern).

[7] *"Radiative semileptonic kaon decays"*, J. Bijnens et al., Nucl. Phys. B396, 118 (1993).

[8] *"T-violating muon polarization in $K^+ \to \mu^+\nu\gamma$"*, C.H. Chen et al., Phys. Rev. D56, 6856 (1997).

[9] *"$\pi \to e\nu, \pi \to e\nu\gamma$ Decays and Related Processes"*, D.A. Bryman, P. Depommier and C. Leroy, Phys. Rep. 88, 152 (1882); see page 157.

7 Appendix: figures

The following graphs are better seen by using a color printer. It takes some time to print.

The units are:

- For the photon,

$$x = 2E_\gamma/M \tag{21}$$

where E_γ is the photon energy.

$$\begin{aligned} 0 &\leq x \leq 1 - r \\ 0 &\leq x \leq 0.427 \\ 0 &\leq E_\gamma \leq 29.8 \text{ MeV} \end{aligned} \tag{22}$$

- For the muon,

$$y = 2E_\mu/M \tag{23}$$

where E_μ is the muon total energy. $E_\mu = T_\mu + m$ (kinetic + rest mass).

$$\begin{array}{rcl} 2\sqrt{r} & \leq y \leq & 1 + r \\ 1.514 & \leq y \leq & 1.573 \\ 105.65 \text{ MeV} & \leq E_\mu \leq & 109.77 \text{ MeV} \\ 0.00 \text{ MeV} & \leq T_\mu \leq & 4.12 \text{ MeV} \end{array} \tag{24}$$

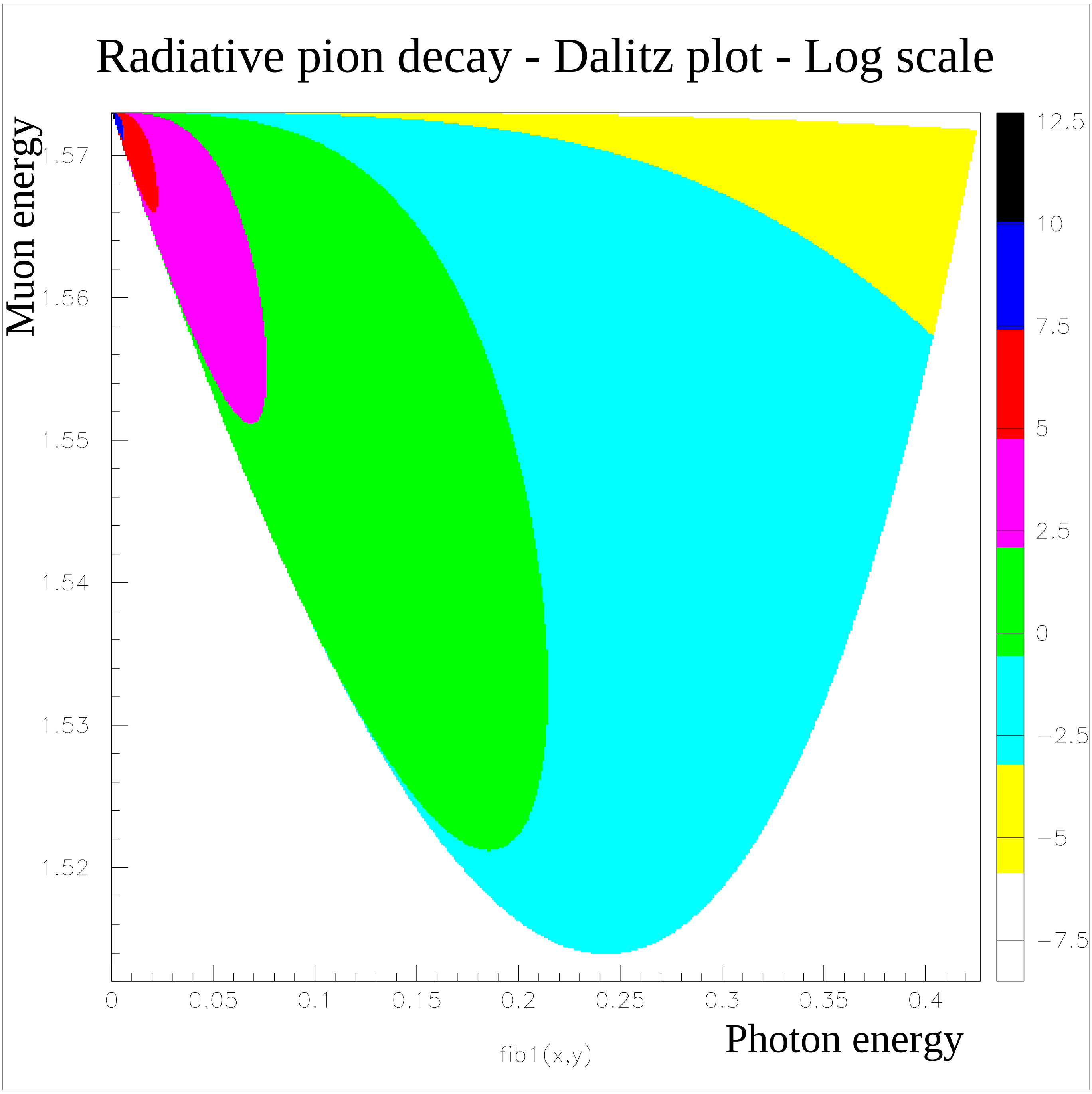

Figure 1: The radiative decay rate. The variation is so large that a log scale is needed. The rate is going down to zero for the maximal muon energy, except for $x = 0$ because of the artificial singularity.

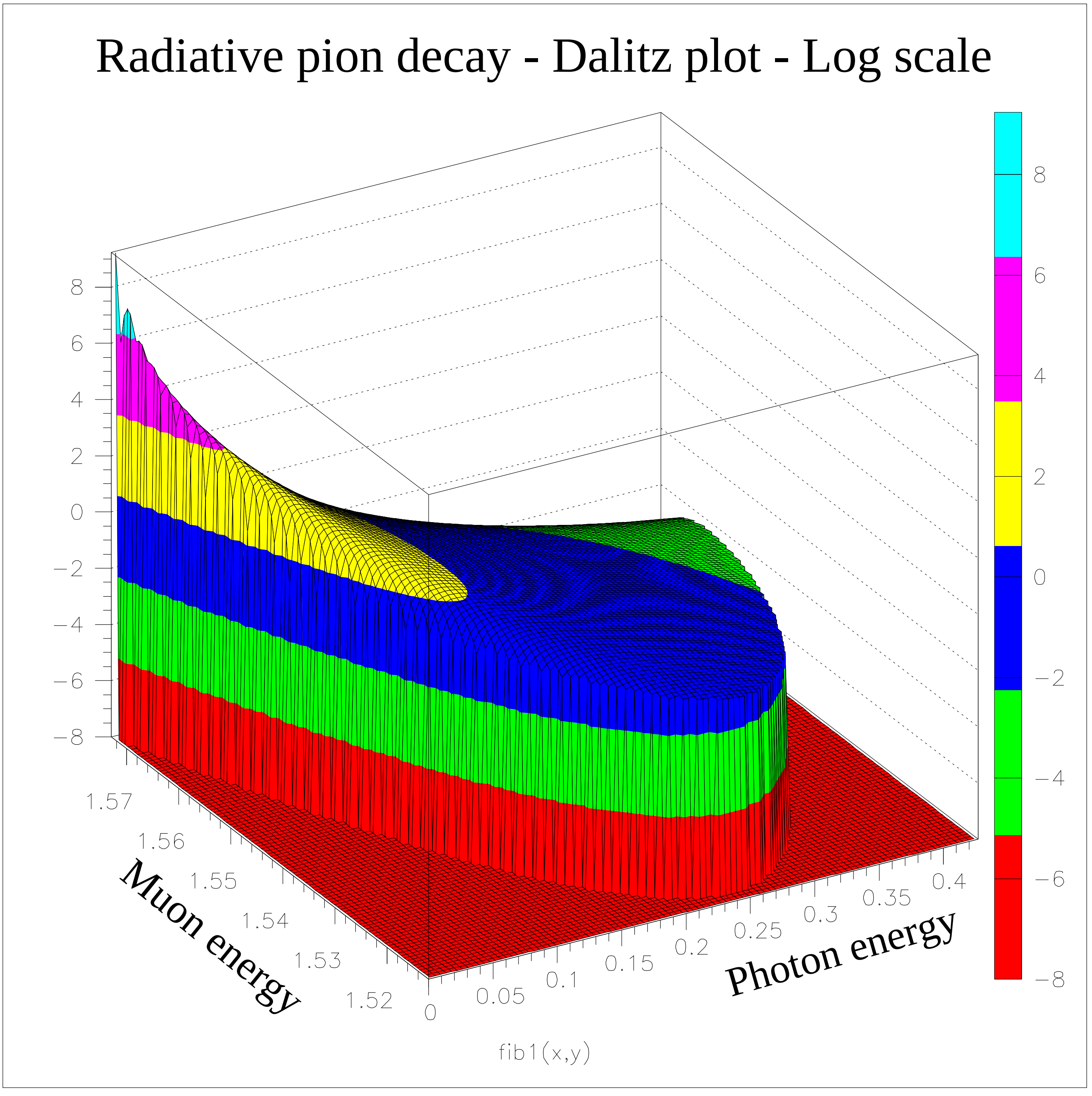

Figure 2: A surf plot of the radiative decay rate.

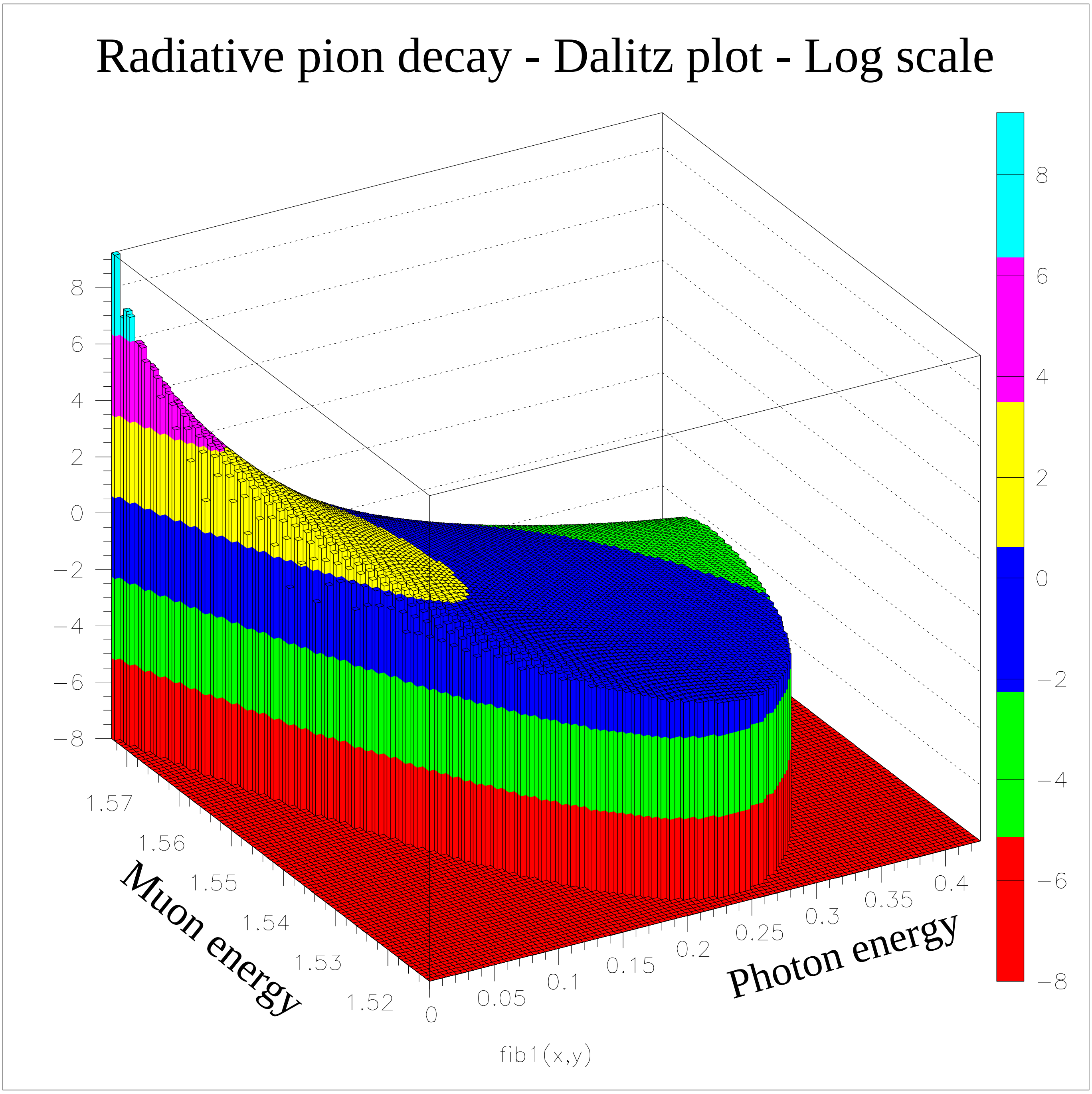

Figure 3: A lego plot of the radiative decay rate.

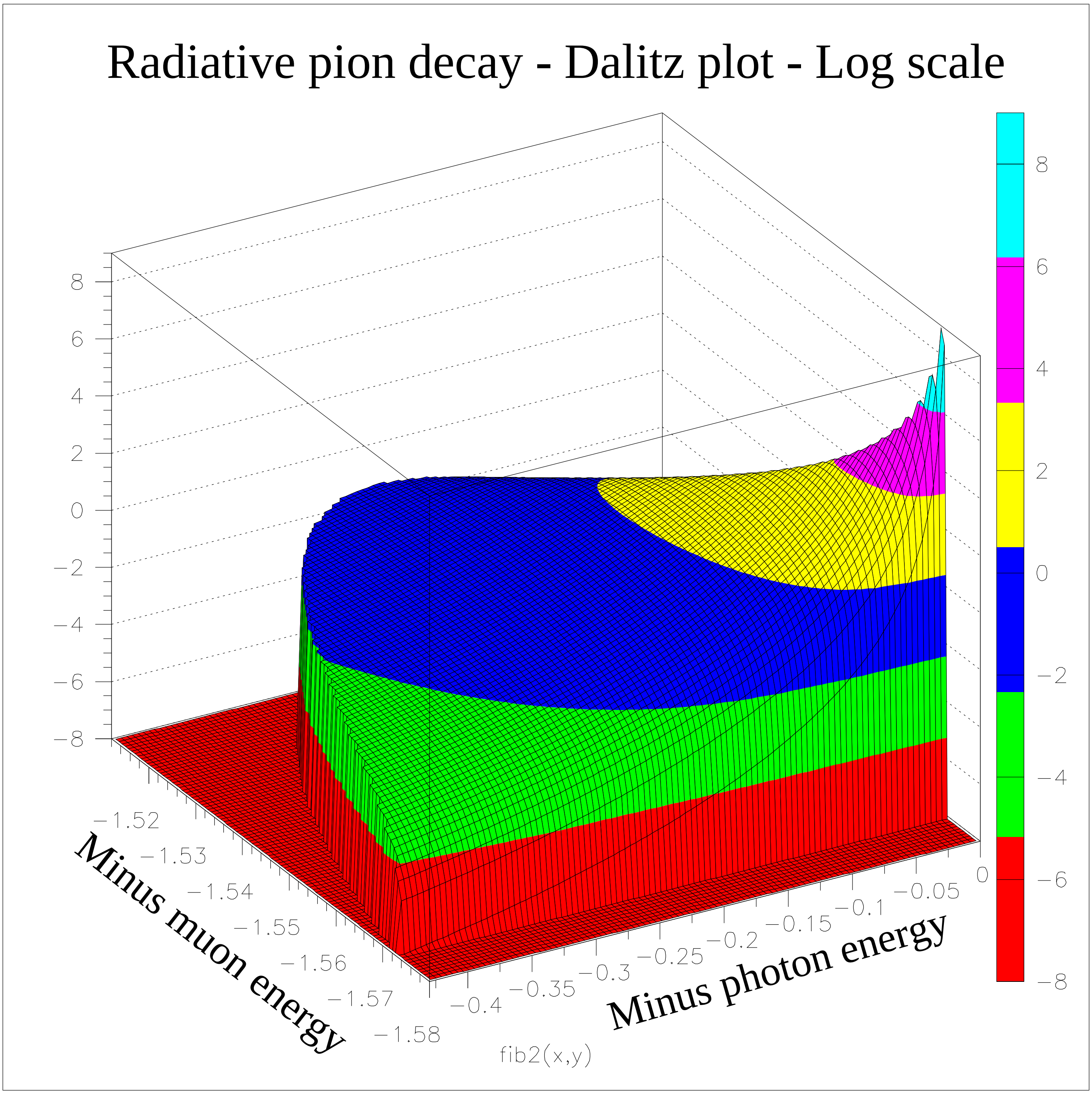

Figure 4: A surf plot of the radiative decay rate. In order to see the opposite side the signs of x and y have been changed.

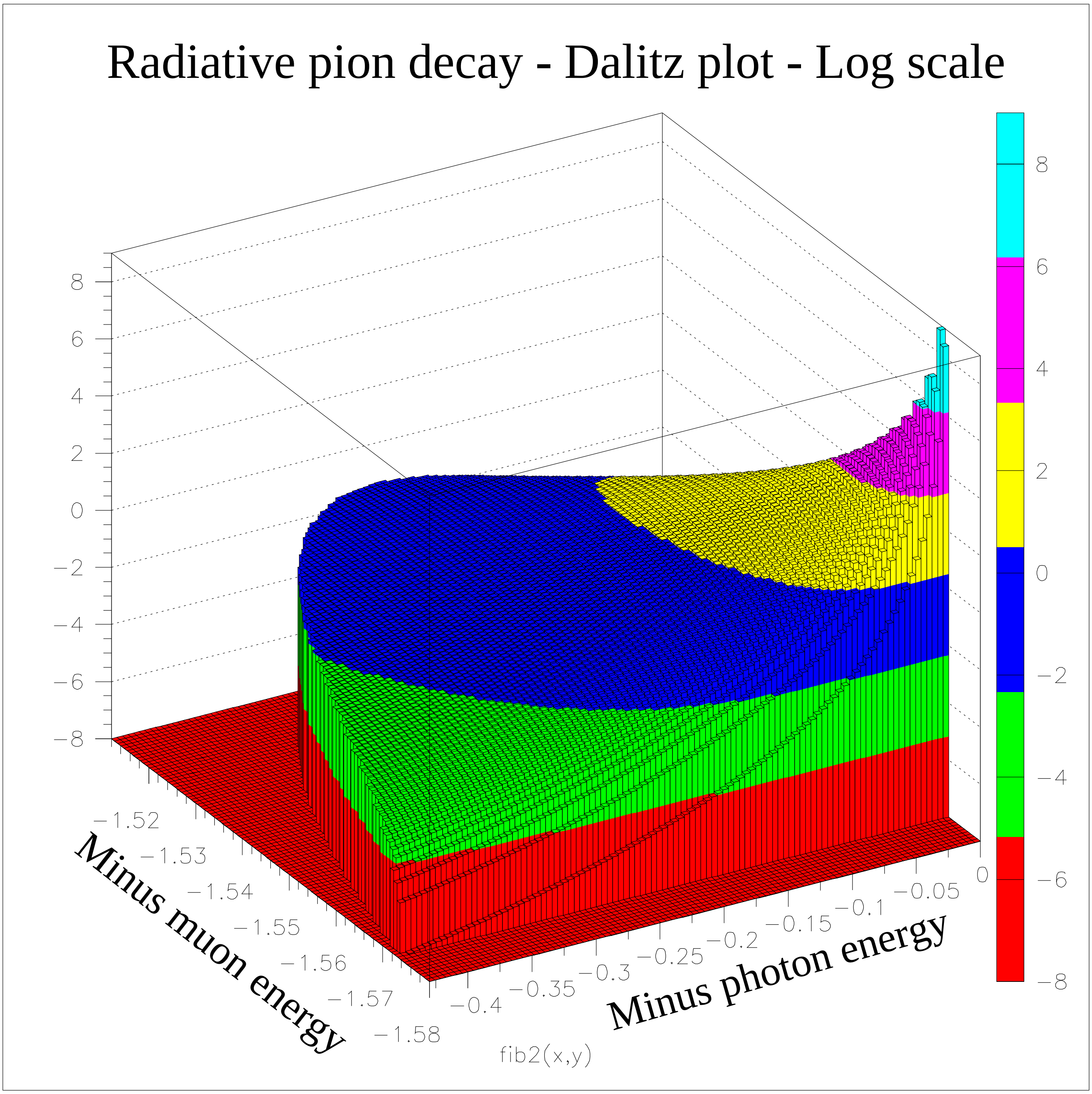

Figure 5: A lego plot of the radiative decay rate. In order to see the opposite side the signs of x and y have been changed.

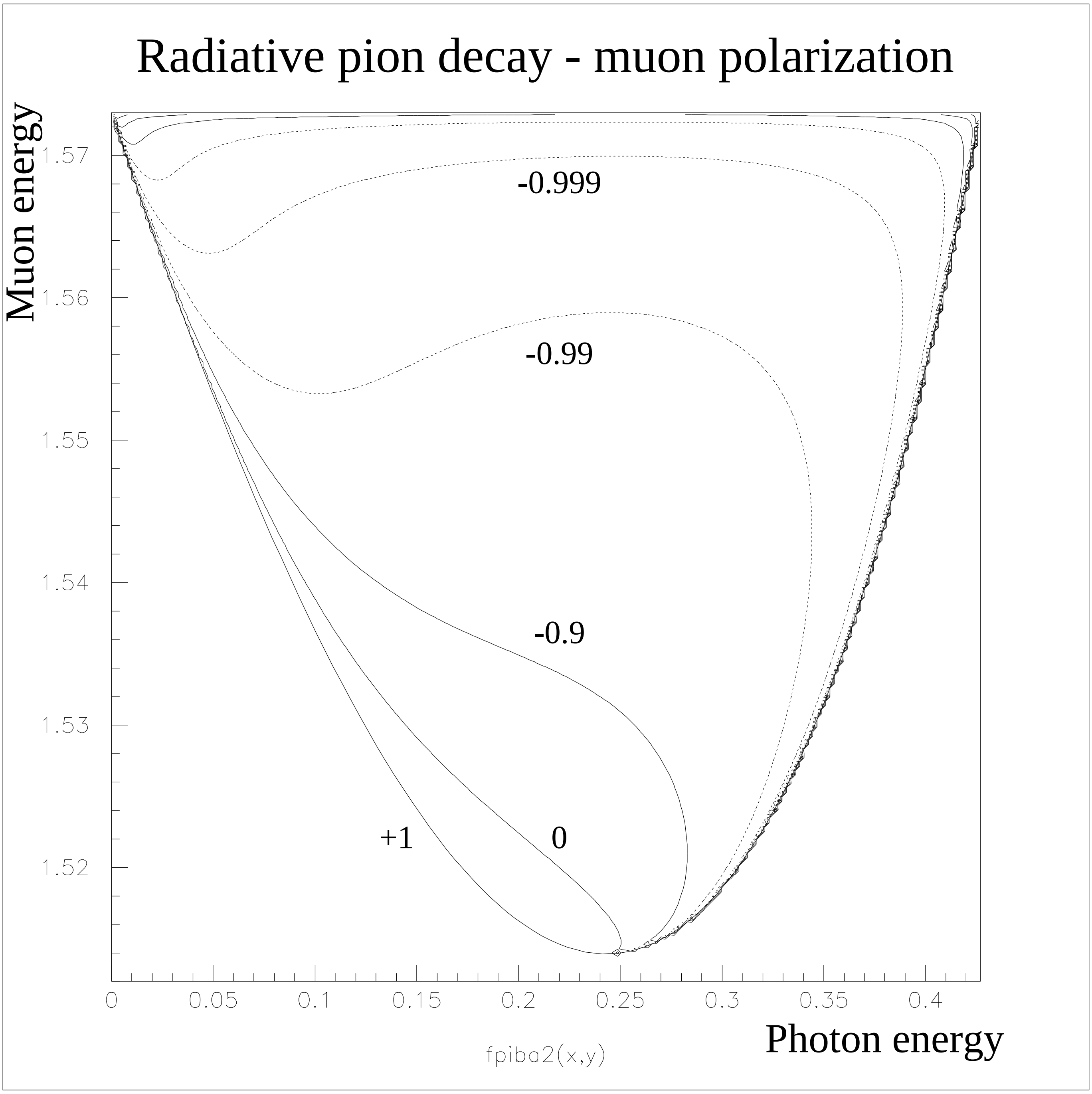

Figure 6: A contour plot for the longitudinal polarization P_L.

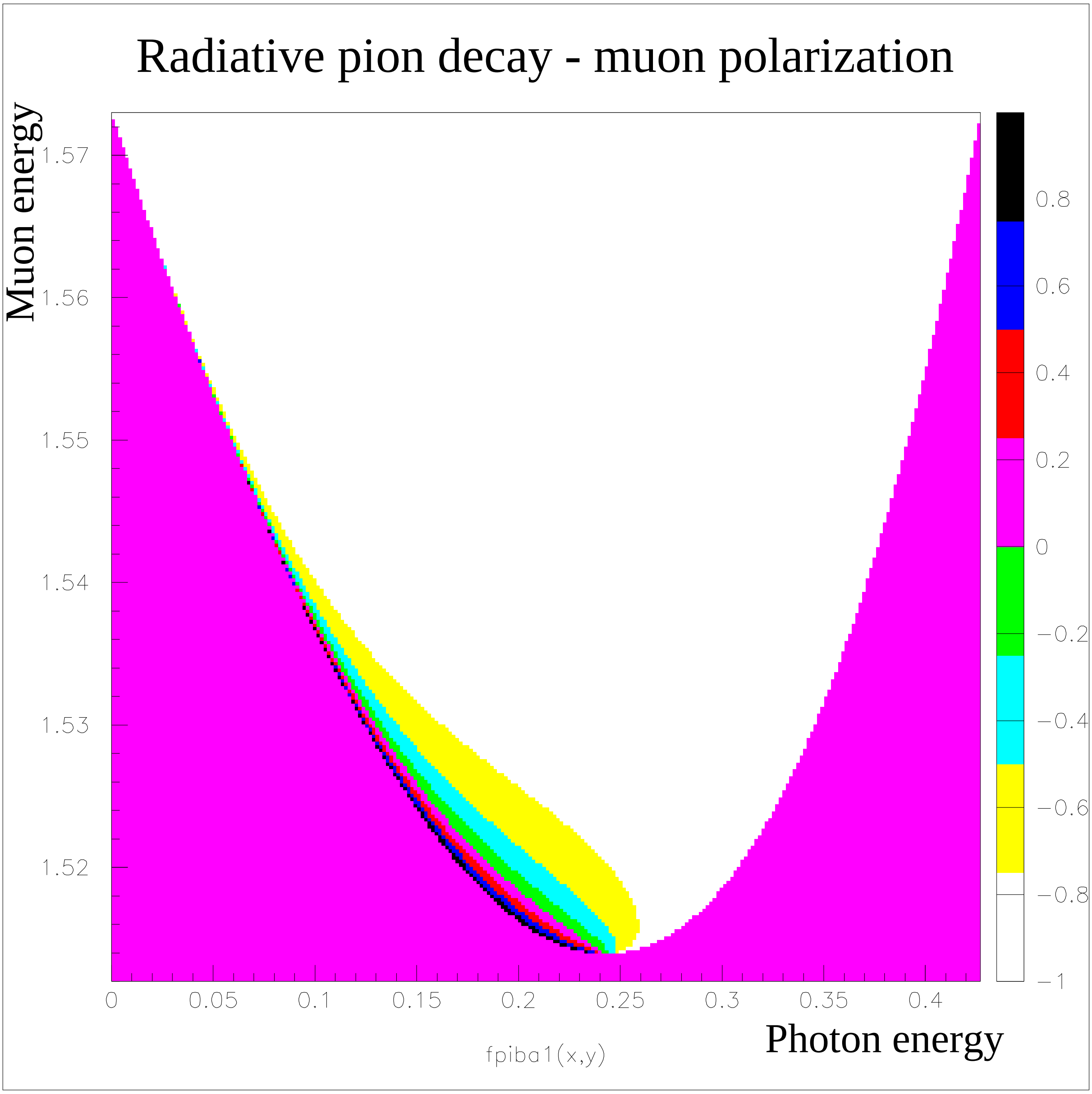

Figure 7: The longitudinal polarization P_L. Again it can be seen that P_L is mostly negative.

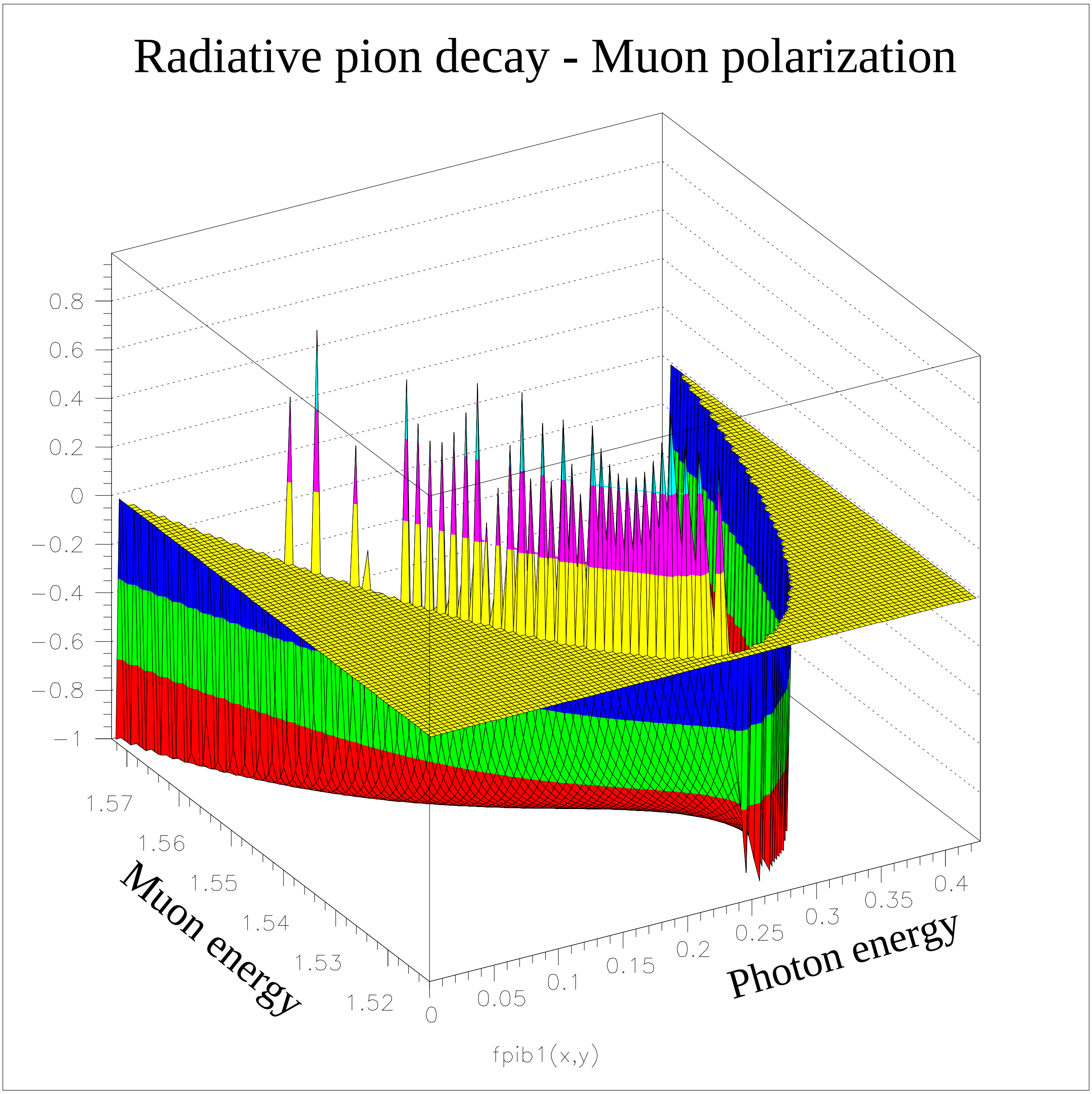

Figure 8: A surf plot of the longitudinal polarization P_L. Positive values of P_L are seen in a small region of the Dalitz plot.

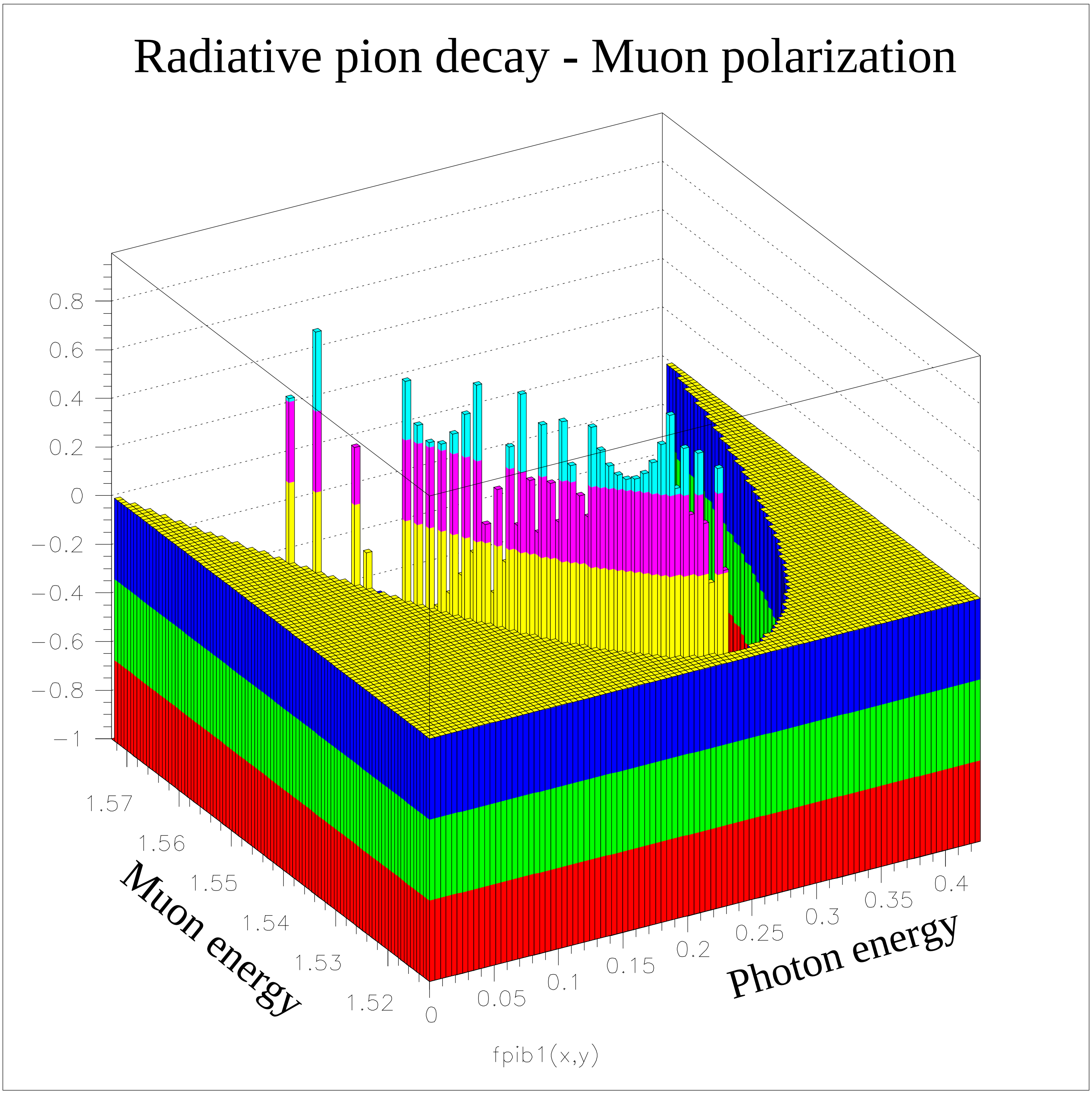

Figure 9: A lego plot of the longitudinal polarization P_L. Positive values of P_L are seen in a small region of the Dalitz plot.

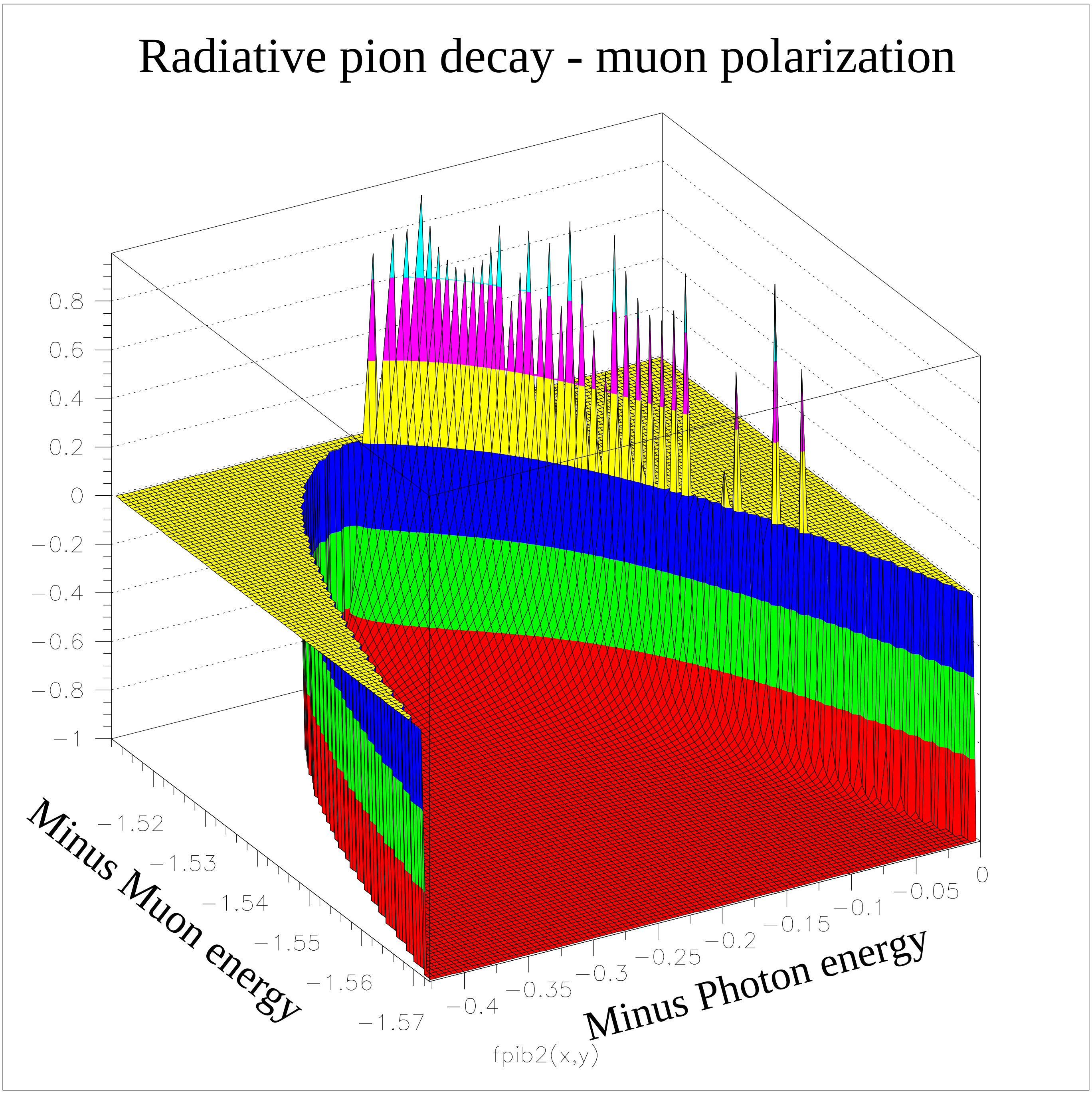

Figure 10: A surf plot of the longitudinal polarization P_L. In order to see the opposite side the signs of x and y have been changed.

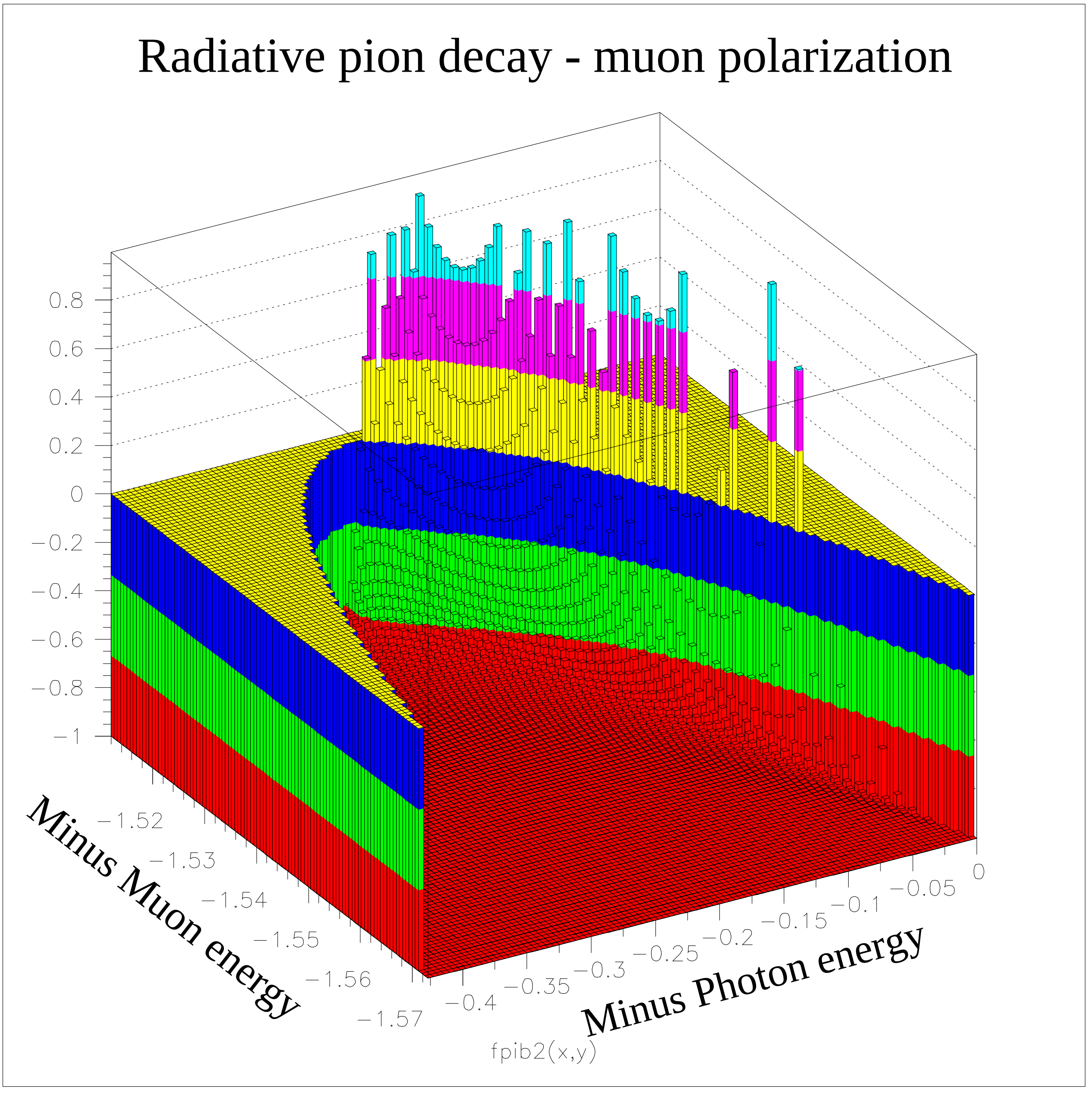

Figure 11: A lego plot of the longitudinal polarization P_L. In order to see the opposite side the signs of x and y have been changed.