The band between 50.0 and 52.8 MeV/c will be depleted by that amount, which is 5%. The band between 45 and 50 MeV/c loses an amount almost equal to what it gains, so that to first order there is no net change in its population. This effect is taken care of later in Sec. VII by estimating its effect on ρ. It is of course surprising that such a large effect is present in the φ_T histograms when the momentum error is so small.

The small slope, less than 1%, at other momenta is due to vertical focussing. The effect also causes a systematic error and is corrected for later in Sec. VII.

The final momentum spectra which were selected by the foregoing are tabulated in Appendix IV for each field. Approximately 80% of the data are rejected because they fall outside of the limits on α' and ϕ_T.

VII. EXPERIMENTAL RESULTS AND DISCUSSION OF THE EXPERIMENTAL ERROR

A. The Internal Consistency of the Data and the Evaluation of ρ

Since each momenta spectrum obtained at a given value of magnetic field permits an independent measurement of ρ, each spectrum was compared separately to the theoretical spectrum. The theoretical spectrum is composed of two parts; the first part is the spectrum for the case $\rho = 3/4$ and $\eta = 0$, and the second part represents the difference between the spectrum for $\rho = 3/4 + \Delta\rho$ and $\eta \neq 0$. The inclusion of corrections is described in Sec. V. Data for the experimental spectra were selected according to Sec. VI-C. The comparison between experimental and theoretical spectra is made by computing χ_j^2, defined by Eq. (40).

$$\chi_j^2(\Delta\rho,\eta) = \sum_{P_i=P_{min}}^{P_{max}} \frac{[\alpha^j N_{th}(P_i) - N_{ex}^j(P_i)]^2}{N_{ex}^j(P_i)} \tag{40}$$

j denotes the magnetic field the data were taken at. j = 6.6 kG, 5.3 kG, and so forth. $N_{ex}^j(P_i)$ are the number of events in the i^{th} momentum interval observed experimentally at the j^{th} field setting. $N_{th}(P_i)$ is the relative density of events in the i^{th} momentum interval for the theoretical spectrum and is chosen to satisfy Eq. (41).

$$\alpha_j \sum_{P_i} N_{th}(P_i) = \sum_{P_i} N_{ex}^j(P_i) \tag{41}$$

For the particular hypothesis $\rho = 3/4$ and $\eta = 0$. χ^2 has been computed for each magnetic field and the results are shown in Table VIII. They show that with the exception of the data taken at 5.3 kG the agreement with the hypothesis is good and hence the data show consistency from one field to another.

TABLE VIII

Values of χ^2 for the Hypothesis ρ=0.750 and η=0.0

Momentum Region (MeV/c)	Magnetic Field (kG)	Degrees of Freedom	Chi Square
52.6-38.6	7.2	35	30
52.6-34.0	6.6	62	60
44.9-27.9	5.3	55	85
44.9-33.6	5.3	39	44
31.6-19.4	4.4	40	43
21.8-13.6	2.6	25	21

The data taken at 5.3 kG do not agree with the hypothesis and the disagreement can further be isolated to a part of 5.3 kG data. The disagreement will be discussed at length later in this section. In addition to computing χ^2 for the hypothesis $\rho = 3/4$ and $\eta = 0$, the most probable value of ρ was found by minimizing χ^2 with respect to $\Delta\rho$. The results of these calculations are presented in Table IX. The data from 2.0 kG were not statistically significant, as they were consistent with all values of ρ from +1 to 0.

TABLE IX

Most Likely Value of ρ When η=0 at Each Magnetic Field

Momentum Region (MeV/c)	Magnetic Field (kG)	Number of Events	Degrees of Freedom (deg)	Chi Square	$\rho\pm$ Statistica Error
52.6-38.6	7.2	11,799	35	30	0.7500±0.022
52.6-34.0	6.6	744,085	62	59	0.7510±0.0024
44.9-27.8	5.3	400,000	53	84	0.7384±0.0053
44.9-33.6	5.3	301,668	38	36	0.748 ±0.0082
31.60-19.4	4.4	85,304	40	43	0.746 ±0.022
21.80-13.65	2.6	13,777	25	21	0.750 ±0.167

The agreement of one field with another is good except for the 5.3 kG data.

Another means of testing agreement of the results obtained at different fields is to make a direct comparison of the spectra shapes where the spectra span a common interval of momenta. The fields were chosen in such a manner that the useful region of momenta at any given field would overlap the useful momenta region of momenta at the next larger and the next smaller field.

For example, at 6.6 kG and 5.3 kG the useful region of momenta for both fields contains the interval between 34.0 and 44.9 MeV/c. The extent of the useful region of momenta at any given field setting is based on criteria described in Sec. VI-D. Since no attempt was made to monitor the muon stopping rate in the target counter, the spectra first must be normalized so that each has an equal number of events in the overlap region. The normalization is done by Eq. (42):

$$\sum_{P_i} [C_j N_j(P_i) - C_{j-1} N_{j-1}(P_i)] = 0 \qquad (42)$$

C_j is the normalization constant for the j^{th} field. Equation(40) is defined to be 1. A χ^2 is computed for each pair of overlapping spectra by Eq. (43)

$$\chi^2_{j,j-1} = \sum_{P_i} \frac{[N^j_{ex}(P_i) - (C_{j-1}/D_j) N^{j-1}_{ex}(P_i)]^2}{N^j_{ex}(P_i) + (C_{j-1}/C_j)^2 N^{j-1}_{ex}(P_i)} \qquad (43)$$

The sum is carried out over the momenta interval common to each pair of spectra. A point-by-point comparison of one overlap region is shown in Fig. 33. The results of the overlap computations are presented in Table X.

TABLE X

Comparison of Overlap Regions

Overlap Region (MeV/c)	No. of Degrees of Freedom	No. of Events in Each Region		χ^2_j, j-1
44.9-34.0 (6.6 and 5.3 kG)	35	438,032	294,782	39.4 ± 8.5
31.6-27.8 (5.3 and 4.4 kG)	14	91,439	40,154	24.7 ± 5.6
21.8-19.4 (4.4 and 2.6 kG)	7	42,244	5,299	7.1 ± 3.7
15.4-13.6 (2.6 and 2.0 kG)	5	2,382	1,096	5.6 ± 3.2

The results show that agreement between the 5.3 kG and 4.4 kG field spectra is poor, and that all other overlaps show good agreement. The overlap between 4.4 and 2.6 kG and 2.6 and 2.0 kG are statistically weak and the results of Table VIII are more useful in establishing agreement since more events are being compared. The three tables can be combined to show that the data between 33.6 MeV/c and 27.8 MeV/c obtained when the field was 5.3 kG is in strong disagreement with all other data and any reasonable theoretical spectrum. The data of Table X show that either the 5.3 kG or the 4.4 kG data are in error. Table VIII shows that only the 5.3 kG data, and only the part between 33.6 and 27.8 MeV/c, reject the theoretical hypothesis $\rho = 3/4$ and $\eta = 0$. Table VIII also shows that all other data can fit the hypothesis $\rho = 3/4$ and $\eta = 0$. Table IX shows that all data except the questioned data fits a spectrum for some value of ρ very well.

The data for the 5.3 kG field fits no value of ρ well. The probability for all these results to be consistent is well beyond reasonable statistical limits. An examination of the difference between the best-fit theoretical spectrum and the experimental spectrum of 5.3 kG shows that there is no obvious systematic departure. There is only a wide scatter of points. In particular, the contribution to χ^2 in the interval 27.8 to 33.6 MeV/c for the hypothesis $\rho = 3/4$ was 48 for 19 degrees of freedom. The contribution to χ^2 for the remainder of the data at this field setting, data from 33.6 to 44.9 MeV/c, was 38 for 36 degrees of freedom. Because the data taken at

4.4 kG include part of the same momenta band as the data in question, there is an independent check that the anomaly is due to a malfunction of the apparatus and not due to some undiscovered property of muon decay. For the foregoing reasons the data between 27.8 and 33.6 MeV/c taken at the 5.3 kG field was not used in any of the subsequent analyses. If the data were included the value of ρ obtained would only decrease by 0.002, but the fit would be very poor.

Because the ϕ_T distributions were not flat a check of whether the spectra were the same was made by comparing the 6.6 kG data taken for $-19^{\circ} < \phi_T < 0$ with the 6.6 kG data taken for $0 < \phi_T < 18.5^{\circ}$. Except for the momentum intervals which include the endpoint, the agreement was within statistics. Each piece of the data measures ρ to $\pm$ 0.004. An effect is expected at the endpoint and was observed as expected. The small difference between the two spectra caused by vertical focussing was not observed because of statistics. At the other fields the quantity of data was smaller and did not permit meaningful comparisons. The conclusion which can be drawn from these checks is that the data is self-consistent except as noted earlier.

Using only the data which are self-consistent the most probable value of ρ was computed by minimizing the sum of the χ_j^2, (defined by Eq. (40)). The sum is carried out over all the field settings with the value of ρ:

$$\rho = 0.7498 \pm 0.0022 \quad \text{when } \eta = 0 .$$

The error is the statistical error and corresponds to the values of ρ where χ^2 has increased by 1 from its minimum value. The value of χ^2 was 188 for 213 degrees of freedom. A plot of χ^2 as a function of ρ is shown in Fig. 34.

In addition to the case $\eta = 0$ and ρ variable, the case $\rho = 3/4$ and η variable was investigated. When $\rho = 3/4$ the most probable value of η is

$$\eta = 0.05 \pm 0.050 \qquad \text{when } \rho = 3/4 \quad .$$

The preceding values for ρ and η have not been corrected for several small systematic errors which were present in the experiment.

B. Experimental Check of Ionization Loss and Bremsstrahlung

Since the largest corrections made to the data were those due to bremsstrahlung and ionization loss, part of the experiment was devoted to checking these corrections. Most of the bremsstrahlung is generated in the 1/8-in. plastic scintillator which is used as the target. By placing a 3/8-in. thick piece of the same type of plastic scintillator between the first chamber and the target counter, the positrons which emerged from the target counter were degraded in energy by additional bremsstrahlung and ionization loss. The 3/8-in. plastic scintillator served no purpose other than to degrade the positron energy, and all detected positron trajectories began in the 1/8-in. target counter. The bremsstrahlung was increased by a factor of 7 and the ionization loss outside of the target by a factor of 20 beyond the normal experimental conditions. Of the 600,000 events of this kind recorded, 120,000 of these were in the useful regions of momentum and solid angle. Two relevant numbers were obtained from the data: the shift in the endpoint, which measured the most probable ionization loss; and a value of ρ, which gives a measure of how well the corrections were made in the momentum interval between 34 MeV/c and 52.8 MeV/c.

The shift in the endpoint was 1.65 MeV/c and is to be compared with the 1.68 MeV/c predicted by the following formula[30]

$$\Delta_p = (0.1537)\left(\frac{\Sigma_Z}{\Sigma_A}\right) D[19.43 + \ln(D/P)] \tag{44}$$

Δ_p is the most probable energy loss in Landau's probability distribution for ionization loss. The shift in the endpoint should correspond closely to the most probable energy loss. The endpoint was defined to be the momentum at which the population of the experimental spectrum had fallen to half the population at 50.0 MeV/c. Experimental results agree with both Δ_p and the shape of the Landau distribution for energies below 15 MeV. Above that energy there are no experiments. In particular the results of Goldwasser, Hanson, and Mills at 15 MeV/c agree with the Landau theory to within a few percent.[30] The value of ρ obtained from this data provides a more direct check of the corrections. If bremsstrahlung had been ignored for the data taken under normal experimental conditions the value of ρ obtained would be 0.005 lower than the correct value. When the 3/8-in. plastic scintillator is added, the value of ρ would be decreased by an additional 0.032 if bremsstrahlung were ignored. The effect of straggling in the ionization loss is to lower ρ by 0.03 if it is ignored when the 3/8-in. plastic is added.

On the basis of including all the corrections for the extra plastic in addition to those normally made, the value of ρ obtained from this data was $\rho = 0.748 \pm 0.008$, which is in excellent agreement with the other data. Since the effects which are corrected for are eight times the statistical error, one can conclude that the corrections for the case without the extra scintillator were correct to at least 0.001 in ρ.

C. An Upper Limit to the Muon Neutrino Mass

Another experimental result which can be obtained from this experiment is an upper limit to the muon neutrino mass. Ignoring radiative corrections a displacement of the endpoint from a value less than[31]

$$P_e = \frac{M_\mu C}{2}\left(1 - \left(\frac{M_e}{M_\mu}\right)^2\right) = 52.826 \text{ MeV/c} \tag{45}$$

52.828

is a sign that the muon neutrino mass is not zero. If ΔP is the discripancy in the endpoint, then

$$M_\nu^{\ 2} \approx 4c^2(\Delta p)(k_\nu) \tag{46}$$

where k_ν is the momentum of the muon neutrino. After averaging over all values of k_ν, the upper limit on M_ν can be put as:

$$M_\nu^{\ 2} < \Delta c^2(\Delta p)(\bar{k}_\nu) \approx c^2(\Delta p)M_\mu \tag{47}$$

$\bar{k}_\nu$ is the average muon neutrino momentum when emitted collinear with the electron neutrino and the electron. The shape of the endpoint can in principle give more information, but it is uncertain by the uncertainty in the experimental momentum resolution. The preceding discussion is based on the assumption that the electron neutrino mass is zero. This mass is known to be less than 200 eV and thus the assumption is reasonable.[32]

Aside from the pulse height correction which has been discussed previously, only two corrections must be made to the spectrum at the endpoint to obtain the endpoint. The experimental endpoint must be corrected for energy loss by ionization in the wrapping of the target counter, spark chamber I, and the vacuum tank. The loss, 0.12 MeV/c, was computed from Eq. (44) and the contribution of each piece of material to the loss is given in Table XI.

TABLE XI

Sources of Ionization Loss
Which Were Not Accounted for in Reconstruction

Material		D (grx10^3/cm^2)	t (cm)	Δp (MeV/c)
Mylar				0.077
Vacuum Windows	0.010-in.	34.9	0.0252	
Two Spark Chamber Windows	0.003-in.	21.2	0.0154	
Inside Counter Wrapping (Aluminized Mylar)	0.001-in.	3.6	0.0026	
Two Aluminum Foil Electrodes	0.001-in.	13.7	0.0051	0.014
1 Layer of Black Polyethylene		8.0	0.008	0.010
4 cm Air		5.3	4.0	0.008
4 cm Neon		3.5	4.0	0.006
Total Energy Loss				0.115

The true endpoint is not an observable property of the spectrum, due to the presence of the corrections outlined in Sec. V-B and Sec. V-C. The endpoint is defined operationally as that momentum at which the spectrum has half the population that the spectrum has at 51.5 MeV/c. This definition clearly depends on 51.5 MeV/c as a reference and the use of a halfway point. The shift of the operational endpoint relative to the true endpoint is obtained by applying the operational definition to the theoretical spectrum. The shift was found to be 0.06 MeV/c. The total correction due to the preceding two effects is 0.18 MeV/c.

The momentum resolution of the spectrometer was determined to be ± 0.16 MeV/c at 52.8 MeV/c, from the width of the falloff. The endpoint was determined at three different fields, the results of which are presented in Table XII.

TABLE XII

Experimental Determination of Muon Spectrum Endpoint

Magnetic Field (kG)	Experimental Endpoint (MeV/c)	Corrected Endpoint (MeV/c)
6.6	52.66 ± 0.02	52.84 ± 0.02
7.2	52.68 ± 0.04	52.86 ± 0.04
7.7	52.66 ± 0.04	52.84 ± 0.04
Average of All Fields ± Variance	52.66 ± 0.02	52.84 ± 0.02

The uncertainty in the endpoint at the 6.6 kG field is due entirely to the uncertainty in the magnetic field. At 7.2 and 7.7 kG there are an order of magnitude fewer events in the falloff and the statistical fluctuations in the populations account for the increased uncertainty in the endpoint. These results differ by several hundredths of 1 MeV/c from previously published values.[33] The difference arises from the fact that the trajectory was not corrected during reconstruction for energy loss near Chamber II and the effect of the magnetic field variations was neglected in the earlier publication. On this basis the mass of the neutrino is $m_\nu < 1.5$ MeV with 90% confidence. The experimental variance is consistent with the estimated error in chamber location and the average magnetic field.

D. Errors in ρ Due to Systematic MisMeasurement of Momentum

The systematic errors of the experiment can be grouped into three main categories:

1. A systematic mismeasurement of momentum,
2. The selection of events in the spectrum with a momentum-dependent bias,
3. Inclusion in the spectrum of electrons which have scattered off obstructions, or particles which did not come from the decay of a muon.

These errors are inherent to all counter experiments as was mentioned in Sec. II. However, because of the use of spark chambers it is possible to reduce the systematic errors by two orders of magnitude in most instances.

A momentum mismeasurement can arise from one of the following: Incorrect measurement of the relative position of the spark chambers, systematic error in sound ranging, and inaccurate knowledge of the magnetic field. In all instances the accuracy of these measurements exceeds the damands of the experiment. The relative position of the chambers is known to 0.005 in., and hence the error on momentum is 1 part in 5000. The systematic errors of the sound ranging do not exceed 0.005 in., on the basis of the measurements discussed in Sec. IV. The effect on ρ due to positron measurements does not exceed 0.0001.

The magnetic field was measured with an NMR, and the accuracy of this device exceeds 1 part in 10^4. The fact that the field was not completely uniform, together with the approximation of a constant field, can introduce two types of error. The first is an error in the endpoint, and the second is a non-linear momentum scale. The field between Chamber I and II for 34 MeV/c was on the average 1 part in 2000 lower than the field for the 52 MeV/c trajectories. The variation of the average field is more sensitive to the angle ϕ_T and it is this dependence which causes the appearance of the ϕ_T histograms in Fig. 30. The stretching of the momentum scale causes the measured value of ρ to be low by 0.0007 ± 0.0003. The error in the estimate is due to the uncertainty in the magnetic field. The value of ρ is sensitive to the exact location of the end-point even when the falloff is not included. If the spectrum is constrained to have its endpoint at the value predicted by Eq. (45), then the value of ρ given in Sec. VII-A must be corrected by subtracting 0.0009 ± 0.0005, since the end-point is 1 part in 3000 beyond the value predicted by Eq. (45).

As evidence that the endpoint measurement was made correctly and independent of the location of the trajectory, the consistency of the endpoint measurements is presented in Table XIII.

TABLE XIII

Positron Spectrum Endpoint in Different Parts of the Spectrometer

Azimuthal Angle at the Target	Endpoint Before Field Correction	Endpoint After Field Correction
(deg)	(MeV/c)	(MeV/c)
-10 ± 5	52.69 ± 0.02	52.68 ± 0.02
0 ± 5	52.66 ± 0.02	52.67 ± 0.02
+10 ± 5	52.62 ± 0.02	52.66 ± 0.02
+20 ± 5	52.56 ± 0.02	52.64 ± 0.02

E. Systematic Errors in ρ Due to Momentum Dependent Selection Criteria

Several of the selection criteria introduced momentum biases in the spectrum. In addition the electronic logic which fired the spark chambers caused a small momentum bias. The sources of the biases are listed as follows:

1. Multiple scattering selection criteria,
2. Momentum-dependent choice of solid angle,
3. Event trigger,
4. Pulse-height selection criterion.

The existence of momentum-dependent selection criteria has been mentioned before in connection with the limits set for multiple scattering. The selection was made as momentum-independent as possible by factoring out the momentum dependence of the multiple scattering. As the multiple scattering is not exactly gaussian and since the results are somewhat sensitive to the exact limits which are set, a systematic error is introduced. The error is estimated to be less than $\pm$ 0.001 in ρ.

A more important cause of a momentum-dependent selection criterion is due to the non-uniform magnetic field. The non-uniform field implies the presence of a radial field which causes vertical focussing. The angle α is no longer correctly computed from the relation:

$$\alpha = \frac{Z_{II} - Z_{I}}{R\ \varphi_{12}} \tag{48}$$

φ_{12} is the arc subtended by the trajectory between Chamber I and II. The effect of the radial field is to displace Z_{II} from the value a simple helical trajectory would have given. If this displacement depends on momentum, the actual interval of α which was chosen also depends on momentum. The following analysis shows how this dependence comes about.

Near the median plane the radial field is $\left(\frac{\partial B_z}{\partial R}\right) Z$ and it follows that the displacement of Z_{II} from the value obtained from a simple helical trajectory is given by

$$\Delta Z = (\varphi_{12})^2 R \left\langle \frac{1}{B_o} \frac{\partial B_z}{\partial R} Z \right\rangle \tag{49}$$

After averaging ΔZ over all azimuthal angles and target positions the relation between the true value of α, α_T, and the measured value α_m, can be written as:

$$\alpha_M = \alpha_T \left(1 + \frac{\pi^2 R}{6B_o}\left\langle\frac{\partial B_z}{\partial R}\right\rangle\right) \qquad (50)$$

The average value of $\frac{1}{B_o}(\partial B_z/\partial R)$ over these angles and target positions is 5×10^{-5}/cm. As it causes the measured value of $|\alpha|$ to be greater than the true value and since the difference is larger for larger momenta the choice of α intervals is momentum-dependent. Substituting the relevant number into Eq. (50) the relation between α_M and α_T is

$$\alpha_M = \alpha_T(1+0.002x) \qquad (51)$$

Using the selection criterion outlined in Sec. VI-C the range of α' was 0.2% smaller at 52.8 MeV/c than it was thought to be, and the range of α' was 0.1% smaller at 34 MeV/c than it was thought to be. The effect is to introduce a bias which favors the low momentum by 0.1%. By using the selection criteria of Sec. VI-C the value of ρ given in Sec. VII-A is low by 0.0012 ± 0.0005 due to the effect of vertical focussing. As mentioned in Sec. VI-C the φ_T histograms are distorted by vertical focussing. One may average Eq. (49) over all momenta and target positions and demonstrate the presence of this effect. The non-uniform field also causes the angle φ to be mismeasured. However, the effects on ρ are negligible in comparison to the preceding case for α.

The principal cause of the momentum bias in the event trigger was positron annihilation in flight. All events for which annihilation took place in the target counter, the spectrometer, or the E1 counter were excluded from the experimental spectrum. Since the fraction of positrons which annihilate varies inversely with the energy, the momentum dependence of the experimental spectrum differs from the positron spectrum at the instant of decay. The fraction of annihilations which were excluded can be estimated from the annihilation cross section $\phi(E)$ into two photons. If E is the positron energy then the probability that the positron will annihilate in flight with an electron into two photons is[34]

$$\phi(E) = \pi r_o^2 \left(\frac{M_e}{E}\right)\left(\log\left(\frac{ZE}{m_e}\right) - 1\right) \qquad E >> M_e \qquad (52)$$

Single photon annihilation is a factor of $(\alpha^2 Z)^4$ smaller and since Z = 6 for carbon, the contribution is negligible.[35] Annihilations in the target counter, E1 counter, the first 1/6 of the E2 counter, and the spectrometer introduced a bias. The total amount of material is 1.23 gm/cm^2 and the fraction of excluded events is

$$f(x) = \frac{0.0044}{x}\left[1 + 0.023 \ \ln x\right] \qquad (53)$$

This bias caused the measured value of ρ to be high by 0.0026 ± 0.0005. The uncertainty is due to the uncertainty in the fraction of E2 which must be traversed before a count is registered in E2.

Another possible source of bias was positron dependence of the spark chamber efficiency. By examining the class of events which missed a single chamber it was possible to determine where the spark would have been from the trajectory determined by the other three chambers. By demanding that the missing spark correspond to a point within the chamber fiducial volume the chamber efficiencies were found to be within 0.1% to 0.4%. The missing spark did have an unusual distribution as the misses tended to cluster at the edges of the chamber. This can be accounted for in part by the fact that a positron can scatter off the steel frame in the inactive region and into the remaining chambers. The reconstructed event will tend to have distribution which extends into the active region. The misses which were not associated with the edges, approximately one-half of the misses, are associated with low momentum trajectories which scatter very badly. The interpretation of these events is that they are muons. There is no evidence that there is a systematic inefficiency in the spark chambers and this systematic error is put at less than 0.001 in ρ.

The double spark criterion in Chamber I together with the pulse height criterion of 0.54 MeV placed a limit on the maximum kinetic energy of the secondary electrons produced by Bhabba scattering.[36] The limit was estimated to be 0.27 MeV. The rejection of all events with secondary electrons of kinetic energy greater than 0.27 MeV introduced a momentum bias since high momentum positrons were more likely to produce energetic secondaries than were low momentum positrons. If ΔE is the cutoff kinetic energy, the probability that a positron of energy E produced a secondary with energy greater than $\Delta E + M_e$.

$$\phi(E) = 2\pi r_o^2 \left[\frac{M_e}{\Delta E+M_e} - \frac{M_e}{E} \left(2\ln\left(\frac{E}{\Delta E+M_e}\right) - \frac{4}{3} + \frac{3(\Delta E+M_e)}{E} \right) \right] \quad (54)$$

The expression was derived from the Bhabba scattering cross section and terms of order $(\Delta E+M_e)^2/E$ have been dropped. In addition to Bhabba scattering in the target, events which scattered in the spectrometer between Chamber I and IV would be rejected for ΔE in excess of a few MeV, as a result of the multiple scattering criteria and double sparks. Taking these contributions together, their effect was to reduce the measured value of ρ by 0.0021±0.0005.

F. Systematic Errors Due to Contamination of the Spectrum by Unwanted Events

There are three types of unwanted events which could have introduced a bias if they had been sufficiently numerous:

1) Positrons from muon decays in the anti-counter,
2) Beam muons which scattered into the apparatus,
3) Positrons which scattered off the vacuum tank walls or other obstructions in the apparatus.

All other types of events, whether caused by a real or accidental trigger, either did not give a reconstructable event or gave an event which did not bias the spectrum. The first category gave a bias because of the undetermined energy loss in the anti-counter. The anti was estimated to be 100% efficient for energy losses in excess of 0.2 MeV. There were two sources of positrons which came from the anti. The beam pion could scatter in the target and into the anti; a subsequent event trigger would have been generated if the positron decay were undetected by the anti, and simultaneously was in accidental coincidence with the muon gate (1%). During the experiment the rate of positrons which came from pions stopped in the anti was found to be one-fifteenth of the normal rate. The anti-efficiency for the positron decays was found to be 95%; as a consequence the probability of a positron coming from the anti in this manner is less than one on ten thousand. The second source of muons stopping in the anti were from pions which stopped in the side of the target counter which faced the anti. Of all the stopped pions 10% produced muons which escaped from this face of the target counter, and of these approximately 10% stopped in a region of the anti-counter resulting in trajectories which would satisfy the selection criteria of Sec. VI. The timing of the anti-counter was arranged so that these events would not give triggers, and it is estimated that 90% of these muons had enough range in the anti to be rejected. The relative rate is less than 1 in 1000 of the good event rate. Since the energy loss of the positron could not have exceeded 0.2 MeV without being detected by the anti, the bias introduced by these events is negligible.

The second type of unwanted events arose from muons in the pion beam which scattered in the target, and subsequently went through all spark chambers and stopped in either E1 or E2 counters. Most of these events did not have enough range to reach E2. The delay of the muon gate, 0.5 μsec, prevented these prompt events from triggering the spark chambers. A muon which stopped in E1 generated a muon gate and the efficiency of E1 and E2 for counting the subsequent decay positron was almost 50%. The accidental coincidence rate between T^- and the coincidence of E1 and E2 was measured to be 5%. The spark chambers were fired by this accidental coincidence and because the chambers were sensitive for 5 μsec, there was a reasonable chance that the muon track would give sparks in all chambers. The event would have reconstructed with a momentum between 35 and 60 MeV/c. These muons must lose a significant fraction of their energy as they traverse the spectrometer. The average value of DR4ACT for a 40 MeV/c muon is +10 mm. Since the scattering limits of the DR4 criterion are ± 3 mm at 40 MeV, 95% of the muons were rejected by this criterion. Since the multiple scattering of 40 MeV/c muons is three times that of 40 MeV/c positrons, the DZ4 and DZ3 scattering criteria rejected an additional factor of three of these events. If all of the accidentals were assumed to be due to beam muons (5%), then the fraction of contamination cannot exceed 1 part in 1,200. The effect is therefore negligible.

The third type of unwanted events were the positrons which scattered off the vacuum tank walls and reconstructed as a lower energy positron. These events triggered the apparatus although their presence in the spectrum is unlikely since the DR4, DZ3, and DZ4 selection criteria eliminated them. An estimate of the fraction of the events that were scatterings can be made by noting that events which scattered off obstructions must have very broad distributions of DZ3 and DR4, and appear as a flat background in the DZ3 and DR4 histograms. It is estimated, when applying only the DZ3 criterion (or the DR4 criterion) and permitting the other scattering limits to be the maximum (± 2 cm at 50 MeV/c), that 0.3% are the events due to scattered positrons. Since DZ3 and DR4 are independent criteria the total number of positron events which had these scatterings is less than 0.1%. This is an overestimate since there should be a tail due to coulomb-scattering and the tails of the DZ3 and DR4 histograms could be attributed to this. The effect is considered to be negligible.

G. Experimental Value of ρ

The results of the preceding analysis, together with the best estimate of the value of ρ, lead to a corrected estimate of $\rho = 0.7503 \pm 0.0026$. The systematic corrections are summarized in Table XIV.

TABLE XIV

Summary of Experimental Systematic Corrections to ρ

Source of Systematic Correction	Best Estimate Of Correction	Uncertainty In Correction
Stretching of Momentum Scale	+0.0007	±0.0003
Uncertainty of the Endpoint	-0.0009	±0.0005
Vertical Focussing	+0.0012	+0.0005
Positron Annihilation	-0.0026	±0.0005
Bhabba Scattering	+0.0021	±0.0005
Spark Chamber Efficiency	---	±0.0010
Contamination by Unwanted Events	---	0.000
Sum of Corrections	+0.0005	±0.0014
Best Estimate of ρ Before Correction	+0.7498	±0.0022
Experiment Value of ρ Including Corrections	0.7503	±0.0026

The data are sufficiently insensitive to η that the systematic corrections are not necessary to the best fit value of η where $\rho = 3/4$. This result is

$$\eta = 0.05 \pm 0.50 \quad \text{when } \rho = 3/4 .$$

VIII. INTERPRETATION OF THE EXPERIMENTAL RESULTS

The results of this experiment can only be interpreted after a definite order for the lepton fields in the interaction hamiltonian have been chosen. The order of the lepton fields cannot be established from the interaction of the leptons alone, since the order can be changed by a Fierz transformation on the hamiltonian without changing the measurable quantities. A discussion of this ambiguity is given in Appendix I. Since muon decay is an example of a weak interaction, the natural ordering to choose is the ordering that follows from charged lepton currents. The fact that neutral lepton currents have never been observed, makes it plausible that the charge retention order, an interaction of neutral currents, is incorrect. Unfortunately, almost all theoretical work that has been done on muon decay has used the charge retention order.

Even after the choice of an order has been made the range of possible interpretations is still too large to handle. One more reasonable restriction that can be placed on the hamiltonian is to set the scalar coupling constant, G_S, to zero. The experiments which permit this choice are discussed in Appendix I. After imposing the preceding two assumptions the experimentally measurable parameters, ρ, δ, ξ, and η, are functions of only four quantities. The most convenient way to express these quantities is to express them in terms of the fundamental couplings G_V and G_T and the extent to which the neutrinos violate lepton conservation in the hamiltonian. The hamiltonian contains Majorana neutrinos and only the dynamics restrict the possible helicities of the neutrinos. The hamiltonian density on which the interpretation of the experiment is based is given in Eq. (55).

$$H_I(x) = \sqrt{8}\, G_V \tilde{\mu} \gamma_\alpha \left(\frac{(1+i\gamma_5)}{2} + \alpha_\mu^V \frac{(1-i\gamma_5)}{2} \right) \nu_\mu \tilde{\nu}_e \left(\frac{(1-i\gamma_5)}{2} + \alpha_e^V \frac{(1+i\gamma_5)}{2} \right) \gamma_\alpha e$$

$$+ \sqrt{8}\, G_T \tilde{\mu} \sigma_{\alpha\beta} \left(\frac{(1+i\gamma_5)}{2} + \alpha_\mu^T \frac{(1-i\gamma_5)}{2} \right) \nu_\mu \tilde{\nu}_e \left(\frac{(1-i\gamma_5)}{2} + \alpha_e^T \frac{(1+i\gamma_5)}{2} \right) \sigma_{\alpha\beta} \quad (55)$$

$\frac{(1+i\gamma_5)}{2}$ and $\frac{(1-i\gamma_5)}{2}$ are the helicity projection operators. Since it is known already that the fraction of the righthanded neutrino which participates in weak interactions is small, the α's which describe this amount are less than one. The connection between this hamiltonian (Eq.(55)), and one described by Eq.(5) is given in Appendix I. Because the α's and G_T/G_V are known to be small, approximate expressions for ρ, δ, ξ, and η can be derived which permit easy interpretation of the measurements. Equations (56) to (59) are derived in Appendix I.

$$\rho = \frac{3}{4}\left(1-(\alpha_\mu^V)^2 - (\alpha_e^V)^2 - \frac{17}{4}\left(\frac{G_T}{G_V}\alpha_e^T\right)^2 - \frac{17}{4}\left(\frac{G_T}{G_V}\alpha_\mu^T\right)^2\right) \quad (56)$$

$$\delta = \frac{3}{4}\left(1-3(\alpha_\mu^V)^2 - 3(\alpha_e^V)^2 - \frac{43}{4}\left(\frac{G_T}{G_V}\alpha_e^T\right)^2 + \frac{51}{4}\left(\frac{G_T}{G_V}\alpha_\mu^T\right)^2\right) \quad (57)$$

$$\xi = -1\left(1+2(\alpha_\mu^V)^2 - 4(\alpha_e^V)^2 - \frac{68}{4}\left(\frac{G_T}{G_V}\alpha_e^T\right)^2 + \frac{26}{4}\left(\frac{G_T}{G_V}\alpha_\mu^T\right)^2\right) \quad (58)$$

$$\eta = \frac{6}{\sqrt{2}}\left\{\left(\frac{G_T}{G_V}\alpha_\mu^T\right)(\alpha_\mu^V) + \left(\frac{G_T}{G_V}\alpha_e^T\right)(\alpha_e^V)\right\} \quad (59)$$

As the formulas have been written it is clear that η is not independent of ρ, δ, ξ. Moreover, $(\alpha_e^V)^2$ is known to be zero to < 1% from the helicity measurements in β decay. $\left(\frac{G_T \alpha_e^T}{G_V}\right)^2$ is similarly restricted to less than 1%. Using these values and the best experimental values for δ and ξ, the following limits may be imposed

$|\alpha_\mu^V| < 0.2$ for simultaneous one standard deviation limit of both δ and ξ

$\left|\frac{G_T}{G_V}\, \alpha_\mu^T\right| < 0.02$ for simultaneous one standard deviation limit of both δ and ξ .

These limits in turn restrict η to

$|\eta| < 0.04$.

As can be seen from this discussion the least established feature of muon decay is the helicity of the muon neutrino. The possibility that the muon neutrino can permit a small amount of lepton non-conserving processes was pointed out by Friedberg.[38] The results of this experiment considerably diminish this possibility as will be shown.

The results of the measurement of the spectrum can be presented in either of two ways

$\rho = 0.750 \pm 0.003$ for $|\eta| < 0.04$

$\eta = 0.05 \pm 0.50$ for $\rho = 0.750$.

A simultaneous fit for ρ and η is uninformative, as the errors on ρ become quite large. Fitting for η alone is likewise uninformative since it only poorly confirms the limits imposed on η by existing measurements of ξ, δ, and ρ. This arises from the fact that the spectrum is very insensitive to η. The only new information that is gained from this experiment is the value of ρ for $|\eta|$ restricted to the values consistent with the measured values of ξ and δ. It follows from Eq. (56) alone that α_{μ}^{V} must be less than

$$|\alpha_{\mu}^{V}| < 0.06 \ .$$

A majorana neutrino which satisfies lepton conservation must be $\alpha_{\mu}^{V} = 0$. If this majorana neutrino is to give results identical to those of a two-component neutrino theory, the neutrino mass must be zero. The results of this experiment can be expressed in terms of the upper limits to α_{μ}^{V} and $M_{\nu_{\mu}}$:

$|\alpha_{\mu}^{V}| < 0.06$ (lepton non-conserving amplitude of the muon neutrino)

$M_{\nu_{\mu}} < 1.5$ MeV (upper limit to the muon neutrino mass).

These results provide improved evidence that the muon neutrino is a two-component neutrino.

Another interpretation of the measured value of ρ can be used to establish a lower limit on the intermediate boson mass. Although this experiment does not establish a better lower limit, it does deserve to be mentioned. The high energy neutrino experiments established a lower limit on the mass between 1.2 BeV and 2.2 BeV, depending on the branching ratio of W decays into pions and W decays into leptons.[24] The Columbia p-p collision experiment established an upper limit on the product of the W production cross section and the branching ratio of W into muons. This limit could be interpreted to establish an upper limit on the boson mass between 2.5 BeV and 6.0 BeV.[25]

This interpretation is sensitive to both production mechanism and the branching ratio. The 1.2 BeV limit established by the measurement of ρ does not depend on the branching ratio and is thus free from the aforementioned problems. The result may be sensitive to a detailed theory of higher orders of weak interactions.

A third conclusion that may be drawn from these results is that radiative corrections are properly calculated in the region $\chi = 0.50$ to 1.00. Between $\chi = 0.95$ and 1.00 the radiative corrections are large as they vary from 5 to 10%, while the population in the spectrum is 10^5 events/MeV/c. Since the agreement is within statistics, the check is good to about 5%. The check over the whole spectrum is somewhat better since the corrections amount to 5% in ρ between $0.3 < x < 0.95$ and the results are in agreement with the theory to 1/3%.

The experimental results are clearly in agreement with the V-A theory as outlined in the Introduction and Appendix I. The experiment is in good agreement with the bubble chamber experiments of Ref. 20f and 20g. The results are in good agreement with a recent experiment done by Sherwood and Telegdi which used wire chambers.[39] Their result was

$$\rho = 0.762 \pm 0.012 \qquad \text{for } \eta = 0.$$

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