Low-Energy Electron Spectrum from the Decay of Unpolarized Muons*

H. GROTCH†
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
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An exact expression for the electron spectrum from the decay of unpolarized muons, including first-order radiative corrections, was previously calculated by Kinoshita and Sirlin assuming a $V-A$ theory of weak interactions. Most results in the literature, however, have been expressed in the limit of small electron mass, this approximation being valid if the electron's energy is greater than about $10m_e$. In this paper the exact spectrum is presented, and other approximations are used to derive simple formulas useful in the ranges $E_e>3m_e$ and $E_e<3m_e$. In the low-energy region, it is shown that the spectrum is dominated by the emission of noninfrared photons from the electron. Our results are compared with the Michel formula (no radiative corrections) for $E_e\leq 0.1M_\mu$.

I. INTRODUCTION

THE electron spectrum from the decay of unpolarized muons, including lowest-order radiative corrections, was first calculated by Behrends, Finkelstein, and Sirlin,[1] using a general parity-conserving four-fermion interaction. They found an important correction to the basic decay spectrum developed by Michel.[2] However, as was pointed out by Berman,[3] the formula of BFS was inconsistent in its treatment of virtual and real photons. To be consistent with the calculation of virtual photon corrections, one must treat the real photons as vector mesons of small mass. Taking this into account, Kinoshita and Sirlin[4] obtained the correction function, which, if added to the results of BFS, gives the exact spectrum valid for all energies.

In view of the fact that all available experimental data have been for relativistic electrons, it has been customary to present the electron spectrum in the approximation in which the electron mass is put equal to zero whenever this does not cause any trouble. However, some attempts are now being made to accurately measure the whole spectrum, including the very-low-energy range, and thus the need has arisen to make use of the exact formula.[5] Since the exact spectrum contains an enormous number of terms, most of which are unimportant at low energies, it would be useful to obtain an approximate formula which is valid in this region.

The purpose of this note is to simplify the exact spectrum by obtaining approximate formulas in the ranges $E_e>3m_e$ and $E_e<3m_e$. We note that in the nonrelativistic limit, the spectrum is dominated by the emission of noninfrared photons from the electron. Our results are compared with those of Michel for $x\leq 0.2$, where $x=2E_e/M_\mu$.

II. EXACT SPECTRUM TO ORDER

The exact spectrum is obtained by adding the correction function of KS to the result of BFS. As an independent check the complete exact spectrum was recalculated and agreed exactly with that obtained above.

The results are expressed in terms of the variables x, θ, ω, $\omega_<$ defined by

$$\cosh\theta=E/m_e=\gamma=M_\mu x/2m_e\equiv x/x_m\,,$$

$$\omega=\ln(M_\mu/m_e)=5.332\,,$$

and

$$\omega_<=\ln(\lambda/m_e),$$

where λ is a small photon mass. We assume, with very little error, $\cosh\omega=\sinh\omega=\frac{1}{2}e^\omega$. We then obtain

$$P(x)dx=(M_\mu^5/192\pi^3)x(x^2-x_m^2)^{1/2}dx\{(|\bar{g}_V|^2+|\bar{g}_A|^2)\times[3-2x-(x_m^2/x)]+(|\bar{g}_A|^2-|\bar{g}_V|^2)3(x_m/x)(1-x)\}\,,$$

where[6]

$$|\bar{g}_{V,A}|^2=|g_{V,A}|^2[1+(e^2/2\pi)(\alpha_{V,A}+b_{V,A})].$$

The terms $\alpha_{V,A}$ and $b_{V,A}$ express the radiative correction to the Michel formula. They are not very pleasant looking functions.

$$\tfrac{1}{2}\alpha_{V,A}=S\pm\frac{\theta}{\sinh\theta}+\frac{\frac{1}{2}\omega\sinh\omega-\frac{1}{3}\theta\sinh\theta}{\cosh\omega-\cosh\theta+\frac{1}{3}(\cosh\theta\pm1)}-2\,,$$

where

$$S=(\theta-F_1)\coth\theta+(1-\theta\coth\theta)(\omega-2\omega_<)$$

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† Present address: Physics Department, The Pennsylvania State University, University Park, Pa. 16802.
[1] R. E. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. **101**, 866 (1956); hereafter referred to as BFS.
[2] L. Michel, Proc. Phys. Soc. (London) **A63**, 514 (1949).
[3] S. Berman, Phys. Rev. **112**, 267 (1958).
[4] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959); hereafter referred to as KS. See Appendix C in particular.
[5] R. H. Hildebrand and V. Telegdi (private communication).
[6] g_V and g_A are defined with a Hamiltonian in the charge retention order $(\bar{e}\mu)(\bar{\nu}_\mu\nu_e)$.

and[7]

$$F_1=L\left(\frac{2\sinh\theta}{e^{\omega}-e^{-\theta}}\right)-L\left(\frac{2\sinh\theta}{e^{\theta}-e^{-\omega}}\right)+(\omega-\theta)\ln\left(\frac{\sinh\frac{1}{2}(\omega-\theta)}{\sinh\frac{1}{2}(\omega+\theta)}\right),$$

$$b_{V,A}=2D'+[Q+(\cosh\omega-\cosh\theta)^2Y][3-2x-(x_m^2/x)\mp3(x_m/x)(1-x)]^{-1}x_m^2/x,$$

$$Q=4(\theta\coth\theta-1)\sinh^2\omega,$$

$$Y=(10/3)(\theta\coth\theta-1)+[(5/3)\cosh\omega\mp1](\theta/\sinh\theta),$$

$$D'=2(\theta\coth\theta-1)(\omega-\omega_<-1)+(\theta\coth\theta)(1-\ln 2\cosh\theta-e^{-\omega}\operatorname{sech}\theta)$$
$$+(2\theta\coth\theta+\sinh\omega/\sinh\theta-1)\ln(1-e^{-\theta-\omega})+(2\theta\coth\theta-\sinh\omega/\sinh\theta-1)\ln(1-e^{\theta-\omega})$$
$$+(\coth\theta)[L(e^{-\theta-\omega})-L(e^{\theta-\omega})+L(\tanh\theta)-L(-\tanh\theta)+\tfrac{1}{2}L(e^{-\theta}/2\cosh\theta)-\tfrac{1}{2}L(e^{\theta}/2\cosh\theta)].$$

It should be mentioned that the calculation of BFS is done with a parity-conserving interaction, whereas this work assumes that parity is not conserved. We do not, however, expect any additional contributions since we are obtaining the spectrum from unpolarized muons.

Our result is expressed in the same form as the Michel formula, the only difference being the change from $|g_{V,A}|\to|\bar{g}_{V,A}|$ caused by the radiative corrections. As pointed out by BFS, the radiative corrections can be considered as having perturbed the coupling constants. This perturbation is energy-dependent and is enormous in the low-energy limit.

III. APPROXIMATION $E_e>3m_e$

In the reports of BFS, Berman, and Kinoshita and Sirlin, the electron mass was set equal to zero whenever this did not cause a spurious divergence. This provides accurate results for $E_e\gtrsim10m_e$. The assumption that the electron mass is negligible is equivalent to setting $\coth\theta=1$. A somewhat better approximation, good to a few percent down to $E_e=3m_e$ is obtained by retaining an additional term in the expansion of $\coth\theta$. We set $\coth\theta=1+1/2\gamma^2=c(\gamma)$ and then find

$$a_{V,A}+b_{V,A}=2R'(x)+6(3-2x)^{-1}(1-x)\ln x+\left(\frac{1-x}{3x}\right)\frac{[c(\gamma)(\omega+\ln x)(5/x+17-34x)+34x-22]}{3-2x-(x_m^2/x)\mp3(x_m/x)(1-x)},$$

where

$$R'(x)=c(\gamma)\left[2L(1)-2L(x)+\omega\left(\frac{5}{2}+2\ln\frac{1-x}{x}\right)+\left(3\ln x+1-\frac{1}{x}\right)\ln(1-x)-2(\ln x)^2\right]-[2\ln(1-x)-\ln x+\omega]-2.$$

Combining these results with $P(x)dx$ in Sec. II we obtain the approximate spectrum.

IV. APPROXIMATION $E_e<3m_e$

The previous result indicates that for small x a single term dominates the sum $a_{V,A}+b_{V,A}$. In fact the sum is very nearly $b_{V,A}$ noninfrared (N.I.). Therefore, we return to the exact expression for $b_{V,A}$ and extract the important term

$$(b_{V,A})_{\text{N.I.}}\cong\frac{4(\theta\coth\theta-1)+(1-x)^2\{(10/3)(\theta\coth\theta-1)+[(5/3)\cosh\omega\mp1]\theta/\sinh\theta\}}{x[3-2x-(x_m^2/x)\mp3(x_m/x)(1-x)]}.$$

In the interest of simplicity we have used both x and θ even though x and θ are related.

When x is very close to x_m

$$(b_{V,A})_{\text{N.I.}}\cong\frac{173(1-x)^2}{x[3-2x-(x_m^2/x)\mp3(x_m/x)(1-x)]}.$$

We notice that

$$\lim_{x\to x_m}(b_V)_{\text{N.I.}}=\infty$$

and

$$\lim_{x\to x_m}(b_A)_{\text{N.I.}}=28.8(1-x_m)/x_m=2995.$$

This means that the perturbation of the coupling constants, due to the inclusion of the radiative corrections, is indeed strong in the low-energy limit.

$$|g_V|^2\to|\bar{g}_V|^2\to\infty|g_V|^2\quad\text{as}\quad x\to x_m$$

and

$$|g_A|^2\to|\bar{g}_A|^2\to4.47|g_A|^2\quad\text{as}\quad x\to x_m.$$

The above infinity should not cause alarm because it is exactly canceled by other terms appearing in the com-

[7] $L(x)\equiv\int_0^x[\ln(1-t)/t]dt$ is called a Spence function. For $x\leq1$, $L(x)=-\sum_{n=1}^{\infty}x^n/n^2$. A study and table of this and related functions is found in K. Mitchell, Phil. Mag. 40, 351 (1949).

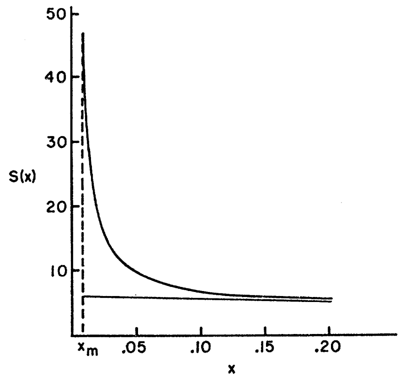

FIG. 1. Comparison with Michel spectrum for $x \leq 0.20$, $|g_V|^2 = |g_A|^2$.
$S(x)$ (upper curve) $= (1/|g_A|^2)[(|\bar{g}_V|^2 + |\bar{g}_A|^2)(3-2x-x_m^2/x) + (|\bar{g}_A|^2 - |\bar{g}_V|^2)(3x_m/x)(1-x)]$.
$S(x)$ (lower curve) $= 2(3-2x-x_m^2/x)$.

plete spectrum. However, we do find, for $|g_A| = |g_V|$,

$$\lim_{x\to x_m} \frac{\text{Radiative contribution to } P(x)}{\text{Nonradiative contribution to } P(x)} = (\alpha/2\pi)(5/36)(M_\mu/m_e)^2 = 6.93.$$

The radiative corrections include a sum of terms with $\mu^- \to e^- + \nu + \bar{\nu}$ and $\mu^- \to e^- + \nu + \bar{\nu} + \gamma$. After cancellation of the infrared divergences, the contributions to the corrected spectrum at low energies come from the uncorrected spectrum of $\mu^- \to e^- + \nu + \bar{\nu}$ and the radiative decay $\mu^- \to e^- + \nu + \bar{\nu} + \gamma_{\text{N.I.}}$ (N.I. denotes noninfrared). The second mechanism is considerably more important at nonrelativistic energies as indicated above.

A qualitative explanation of the importance of single-photon emission was obtained by Kinoshita and Sirlin. Let $N_0(E_e)$ and $N(E_e)$ denote the number of electrons in some small interval about E_e in the uncorrected and corrected spectrum, respectively. Also let $N_0(>E_e)$ be the number in the uncorrected result with energy greater than E_e. If we suddenly turn on the radiative corrections, most of the electrons will be unaffected. However, if E_e is small, $N_0(>E_e) \gg N_0(E_e)$, and, therefore, the number of electrons which radiate sufficient energy to bring them into $N(E_e)$ may well exceed $N_0(E_e)$. In other words, for small E_e, $N(E_e) \cong N_0(E_e) + fN_0(>E_e)$, where f denotes the fraction of the electrons of the uncorrected spectrum which dropped to energy E_e by photon emission (noninfrared). Even though f is small, $N_0(>E_e)$ can be large enough and $N_0(E_e)$ small enough so that $fN_0(>E_e) > N_0(E_e)$.

The size of the radiative correction at low energies makes one wonder whether it is necessary to include higher-order corrections. Explicit examination of the two-photon emission indicates that this will not be important. This should be understandable in terms of the argument above for single-photon emission.

V. COMPARISON WITH MICHEL FORMULA

For comparison we get $|g_V| = |g_A|$ and plot, in Fig. 1,

$$S(x) = \frac{192\pi^3 P(x)}{M_\mu^5 |g_V|^2 x(x^2 - x_m^2)^{1/2}} = (1/|g_V|^2)[|\bar{g}_V|^2 + |\bar{g}_A|^2)(3-2x-x_m^2/x) + (|\bar{g}_A|^2 - |\bar{g}_V|^2)3(x_m/x)(1-x)]$$

as a function of x for $x \leq 0.20$. This is shown along with the Michel result which is simply obtained by replacing $|\bar{g}_{V,A}|^2$ by $|g_{V,A}|^2$. We easily see the importance of the radiative term at low energies.

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