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ELECTRON POLARIZATION IN POLARIZED MUON DECAY: RADIATIVE CORRECTIONS

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Abstract: We calculate the first-order radiative corrections to the decay spectrum of polarized muons, for the case where the electron polarization is also observed. The significance of the results for further studies on muon decay is discussed.

1. Introduction

Muon decay, being the only leptonic process on which precision measurements can be performed to date, is of fundamental importance for the theory of weak interactions. In spite of many beautiful and precise experiments, the "V-A" interaction is far from being clearly established at present. This is illustrated, for example, by Derenzo's analysis [1] of the world data for the Michel parameters ρ, η, ξ, δ and h, in terms of the general four-fermion interaction. He finds that, although a fit to the data favours the "V-A" coupling, it is still compatible with surprisingly large admixtures of scalar, pseudo-scalar and tensor interaction, and/or with a queer combination of vector and axial vector couplings.

As long as there is little hope of identifying the neutrinos (or to establish the masslessness of the muon neutrino), further precision measurements must be made on the decay electron. Here the situation can be improved at least in two ways. First, through a better measurement of the η parameter (which determines the shape of the decay spectrum at low electronic energies). Second, through measurements involving the *polarization* of the electron, such as a direct search for a deviation of the longitudinal polarization from the expected value -1 (for μ^- decay) [2], or a detailed determination of those parts of the decay spectrum which depend on the electron polarization. These experimental quantities contain combinations of the coupling constants, part of which are independent of the Michel parameters. With the advent of the "meson factories" such measurements, which obviously require high intensities, should become feasible with the necessary precision.

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In this paper we present the results of our calculation of the first order radiative corrections to those parts of the decay spectrum which depend on the electron's polarization. To the best of our knowledge these corrections have not been given before. Radiative corrections to muon decay, with and without polarization of the muon but for unpolarized electrons, have been calculated correctly by Berman [3], by Kinoshita and Sirlin [4], and by Källén [5]. The limiting case of small electron energies, i.e. the corrections to the η parameter, have been studied by Grotch [6].

As we shall see below, the extension to the case where the polarization of the decay electron is also observed is not simple, in spite of the fact that a remarkably simple result emerges from our calculation. In our treatment of radiative corrections we have chosen to start from Källén's work to which we shall make extensive reference. The interested will find many more details in his paper. In sect. 2 we briefly indicate the modification of the electromagnetic correction to the muon decay matrix element when the electron spin is not summed over. In sect. 3 we calculate the effect of real Bremsstrahlung on the polarization dependent decay spectrum. Our main result is shown in eq. (11). In sect. 4 we discuss the importance of the results for the questions raised above.

2. Electromagnetic correction to decay matrix element

For this piece of the radiative corrections the extension to the case of polarized electrons is straightforward. Starting e.g. from Källén's eq. (3.18), and neglecting small terms of the order of the electron mass, one finds for the specific case of the decay $\mu^- \to e\,\bar{\nu}_e \nu_\mu$, and for "V-A" interaction, that the differential decay probability is

$$\frac{\mathrm{d}\Gamma^{(1)}}{\mathrm{d}x\,\mathrm{d}\cos\theta} \simeq \frac{G^2 m^5}{384\,\pi^3}\, x^2 \Big\{ (1+2F_1(x))[(3-2x)+\cos\theta\,(1-2x)] - (F_2(x)+F_3(x))\,x\,(1-\cos\theta) \Big\} (1-\cos\varphi). \tag{1}$$

Here and below m and μ denote the muon and the electron mass, respectively. x is the standard dimensionless variable,

$$x = \frac{E}{W} \quad \text{with} \quad W = \frac{m^2+\mu^2}{2m} \simeq \tfrac{1}{2}m, \tag{2}$$

E being the electron energy. θ is the angle between the muon spin and the momentum of the electron $\boldsymbol{p}$, φ the angle between the electron spin and $\boldsymbol{p}$, both defined in the muon's rest frame. The quantities $F_1(x)$, $F_2(x)$ and $F_3(x)$ are defined in eqs. (3.19) and (3.21) of ref. [5]. $F_1(x)$, in particular, contains the infrared-divergent term

$$\Lambda(x) = \ln\left(\frac{m\mu}{\Lambda^2}\right)\left[1-\ln\left(\frac{m}{\mu}x\right)\right], \tag{3}$$

(with Λ the fictitious photon mass) which must drop out of the final result. As is obvious from eq. (1), these corrections do not yet modify the longitudinal polarization of the electron:

$P_l = -1$, independent of x and θ.

3. Internal bremsstrahlung

The calculation of internal bremsstrahlung

$$\mu^- \to e^- \bar{\nu}_e \nu_\mu \gamma, \tag{4}$$

and the integration over the allowed photon momenta, in the case where the electron polarization is included, is considerably more involved than when the spin average is taken. Let us write the differential decay probability as

$$\frac{\mathrm{d}\Gamma^{(b)}}{\mathrm{d}x\,\mathrm{d}\cos\theta} = -\frac{e^2G^2 2m^2 x}{384\pi^3\,(2\pi)^3}\int\frac{\mathrm{d}^3k}{2\omega}\,\Theta((Q-k)^2)\,\{A^{(\mathrm{unpol})}+A^{(\mathrm{pol})}+B^{(\mathrm{unpol})}+B^{(\mathrm{pol})}\}, \tag{5}$$

where the step function keeps track of the kinematically allowed domain of the photon momentum $\boldsymbol{k}$, for fixed electron energy. The expressions $A^{(\mathrm{unpol})}$ and $B^{(\mathrm{unpol})}$ are given in eqs. (4.16a) and (4.16b) of ref. [5] and are not repeated here. For the new expressions $A^{(\mathrm{pol})}$ and $B^{(\mathrm{pol})}$ we find *

$$\begin{aligned}
A^{(\mathrm{pol})} \simeq\; & (p\zeta_e)\, m^2 \left[\frac{Qk}{pk\,qk} - \left(\frac{p}{pk}-\frac{q}{qk}\right)^2\right] \\
& + k\cdot s_e\,\mu m \left[\frac{pq}{(pk)^2} + \frac{Qk}{(pk)^2} + \frac{m^2}{(qk)^2} - \frac{Qk+pq+m^2}{pq\,qk}\right] \\
& + (p\zeta_\mu)(p\zeta_e)\, m \left[\left(\frac{p}{pk}-\frac{q}{qk}\right)^2 \frac{Q^2+2pq-2Qk}{pq} - 4\frac{Qk}{pk\,qk}\right] \\
& + (p\zeta_\mu)\, k\cdot s_e \mu m^2 \left[\frac{3}{pk\,qk} - \frac{1}{(pk)^2} - \frac{2}{(qk)^2}\right] \\
& + k\cdot s_\mu\,(p\zeta_e)\, m \left[\frac{m^2}{pk\,qk} - \frac{m^2}{(qk)^2} - 2\frac{\mu^2}{(pk)^2}\right] \\
& + k\cdot s_\mu\, k\cdot s_e\, \mu m^2 \left[\frac{1}{(pk)^2} + \frac{1}{(qk)^2} - \frac{2}{pk\,qk}\right],
\end{aligned} \tag{6}$$

* We use the metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

and similarly

$$B^{(\mathrm{pol})} \simeq 2(Q^2 - 2Qk)\left\{(p\zeta_e)\left[\frac{Qk}{pk\,qk} - \left(\frac{p}{pk} - \frac{q}{qk}\right)^2\right]\right.$$
$$+ k\cdot s_e\frac{\mu}{m}\left[\frac{pq}{(pk)^2} + \frac{qk}{(pk)^2} - \frac{pq+m^2}{pk\,qk} + \frac{m^2}{(qk)^2} + \frac{1}{qk}\right]$$
$$-(p\zeta_\mu)(p\zeta_e)\frac{m}{pq}\left(\frac{p}{pk} - \frac{q}{qk}\right)^2 + (p\zeta_\mu)\,k\cdot s_e\,\mu\left(\frac{1}{(pk)^2} - \frac{1}{pk\,qk}\right)$$
$$+ k\cdot s_\mu\,(p\zeta_e)\,m\left(\frac{1}{(qk)^2} - \frac{1}{pk\,qk}\right) \tag{7}$$
$$\left.- k\cdot s_\mu\,k\cdot s_e\,\mu\left(\frac{1}{(pk)^2} + \frac{1}{(qk)^2}\right)\right\}.$$

In these expressions the notations are as follows: q, p, k: momenta of muon, electron, and photon, respectively; $Q = q-p$. ζ_μ: muon polarization vector; ζ_e: electron polarization (in its rest frame); and

$$s_\mu = (0;\zeta_\mu); \quad s_e = \left(\frac{1}{\mu}p\zeta_e;\zeta_e + \frac{p\zeta_e}{\mu(E+\mu)}\,p\right),$$

all other symbols having been explained before.

The expressions (6) and (7) are written in the muon's rest frame. Terms which after integration over k are of order O (μ/m) have already been omitted. In eq. (5) a number of new integrals have to be calculated which do not occur in the case where one sums over the electron spin orientations. The most complicated of them are tensor integrals of the type

$$\int\frac{d^3k}{2\omega}\,\Theta((Q-k)^2)\,k_\alpha k_\beta \ldots \,.$$

These are most easily calculated by expressing them in terms of tensors formed with p_α, q_α and $g_{\alpha\beta}$, keeping then track of the orthogonality $ps_e = 0 = q\cdot s_\mu$.

The result of the integration can be written in the form

$$\frac{d\Gamma^{(b)}}{dx\,d\cos\theta} \simeq \frac{G^2m^5}{384\,\pi^3}\,\frac{\alpha}{\pi}\left\{B_0(x) + B_1(x)\cos\theta\right.$$
$$\left.+ C_0(x)\cos\varphi + C_1(x)\cos\theta\cos\varphi\right\}. \tag{8}$$

Here, the functions $B_0(x)$ and $B_1(x)$ stem from the terms $A^{(\mathrm{unpol})}$ and $B^{(\mathrm{unpol})}$ and have been given before, (cf. eqs. (4.29) (a) and (b) of ref. [5]). Both contain the infrared divergency, $\Lambda(x)$, eq. (3), which cancels out in the sum of the decay rates (1) and (8). The new functions $C_0(x)$ and $C_1(x)$ are obtained from $A^{(\mathrm{pol})}$ and $B^{(\mathrm{pol})}$, eqs. (6) and (7). It is obvious from eq. (1) that C_0 and C_1 must contain the same infrared-divergent terms, with opposite sign however, as B_0 and B_1, respectively. This is indeed born out by the calculation. The finite terms, however, need not be the same in either pair of functions.

After a lengthy and tedious calculation we find the following remarkably simple results

$$C_0(x) = -B_0(x) + \tfrac{1}{6}(1-x)^2\,(5-2x), \tag{9}$$

$$C_1(x) = -B_1(x) - \tfrac{1}{6}(1-x)^2\,(2x+1). \tag{10}$$

The new functions $C_0(x)$ and $C_1(x)$ thus differ from $-B_0(x)$ and $-B_1(x)$, respectively, only by a polynominal in x. All finite logarithmic terms such as $\ln(1-x)$ and $\ln(xm/\mu)$ are the same, but for the sign.

Taking the sum of the differential decay rates (1) and (8) and collecting all results obtained so far, one obtains the observable differential decay rate

$$\frac{d\Gamma^{(1)} + d\Gamma^{(b)}}{dx\,d\cos\theta} \simeq \frac{G^2m^5}{384\,\pi^3}\left\{\left[x^2(3-2x) + f_c(x) + \cos\theta\,(x^2(1-2x)\right.\right.$$
$$\left.\left.+ f_\theta(x))\right](1-\cos\varphi) + \frac{\alpha}{6\pi}(1-x)^2\,[(5-2x) - \cos\theta\,(2x+1)]\cos\varphi\right\}, \tag{11}$$

where $f_c(x)$ and $f_\theta(x)$ are the well-known [4, 5] functions

$$f_c(x) = \frac{\alpha}{2\pi}x^2\left\{2(3-2x)\,R(x) + \frac{1-x}{3x^2}\,[(5+17x-34x^2)\ln\frac{m}{\mu}x\right. \tag{12}$$
$$\left.-22x + 34x^2] - 3\ln x\right\},$$

$$f_\theta(x) = \frac{\alpha}{2\pi}x^2\left\{2(1-2x)\,R(x) - \frac{1-x}{3x^2}\left[(1+x+34x^2)\ln\frac{m}{\mu}x\right.\right.$$
$$\left.\left.+ 3 - 7x - 32x^2 + \frac{4(1-x)^2}{x}\ln(1-x)\right] - \ln x\right\}, \tag{13}$$

with the abbreviation

$$R(x) = \left(\ln\frac{m}{\mu}x - 1\right)\left(2\ln\frac{1-x}{x} + \frac{3}{2}\right) + \ln(1-x)\left(\ln x + 1 - \frac{1}{x}\right)$$
$$-\ln x + 2\,L_2(x) - \tfrac{1}{3}\pi^2 - \tfrac{1}{2}, \tag{14}$$

and $L_2(x)$ being Euler's dilogarithm [7].

4. Discussion

We now discuss the relevance of our result (11) for the problems raised in the introduction.

(i) Polarization dependent spectra. For sufficiently large values of the electron energy, say $x \gtrsim 0.25$, the second term in the curly brackets of eq. (11) is small compared with the other radiative corrections $f_c(x)$ and $f_\theta(x)$. The spectrum which multiplies $\cos\varphi$, in particular, thus will have essentially the same corrections as the term independent of the spins. For the case of the general four-fermion interaction, and without radiative corrections, the isotropic part of the spectrum and the $\cos\varphi$ term are written as

$$6x^2 \{(1-x) + \tfrac{2}{9}\rho(4x-3) - \xi' \cos\varphi\, [1-x + \tfrac{2}{3}\delta'(4x-3)]\} \,. \tag{15}$$

For "V-A" interaction one has, of course, $\rho = \frac{3}{4}$; $\xi' = 1$; $\delta' = \frac{1}{4}$. In the case of pure "V-A" interaction, but with radiative corrections included, one can introduce a set of effective parameters ρ_{eff}, ξ'_{eff} and δ'_{eff}, following the method proposed by Kinoshita and Sirlin [4]. Define a normalized spectrum

$$S(x) = \frac{2}{1+F(0)} [x^2(3-2x) + f_c(x)]\,(1-\cos\varphi)\,, \tag{16}$$

with

$$F(x) = 2\int_x^1 f_c(x)\,dx,$$

and approximate $S(x)$ with a Michel formula (15) containing these effective parameters. One then sees immediately that ξ'_{eff} (which one would determine by integrating over x), will remain equal to one, while $3\delta'_{\text{eff}} \simeq \rho_{\text{eff}}$. It is found that radiative corrections reduce the Michel parameter ρ by about 5% [4]. We thus expect a similar reduction of δ'_{eff}.

An analogous discussion applies to the two terms containing $\cos\theta$ and $\cos\theta\,\cos\varphi$.

(ii) Longitudinal polarization of the electron. The longitudinal polarization of the electron is found to be, from eq. (11),

$$P_\ell = -1 + \frac{\alpha}{6\pi}\left(\frac{1-x}{x}\right)^2 \frac{5-2x-\cos\theta\,(2x+1)}{3-2x+\cos\theta\,(1-2x)} \,. \tag{17}$$

Write this as

$$P_\ell \equiv -1 + \delta_P(x,\theta).$$

The deviation δ_P from -1 is indeed very small. For $\theta = 180°$, where the effect is largest, one has

for $x = 0.5$, $\quad \delta_P \simeq 1.2 \times 10^{-3}$,

for $x = 0.25$, $\quad \delta_P \simeq 1.1 \times 10^{-2}$.

In view of the smallness of δ_P one might wonder whether the terms of order $O(\mu)$ should not be kept in the uncorrected spectrum giving also a deviation of P_ℓ from -1. An explicit calculation shows that additional deviation to be smaller, by about one order of magnitude, than the radiative correction δ_P. These terms thus can safely be neglected, except for very low electron energies. For low energies the formula (17) is, of course, not valid. In that region one must keep the well-known terms $O(\mu)$ of the uncorrected spectrum, which appears in the denominator of eq. (17). The numerator of δ_P can be left as it stands, however, even for fairly small values of x.

Our result (17) shows that P_ℓ should be -1 to within say 1% or better (for $x \gtrsim 0.25$), if the pure "V-A" coupling applies *. Any deviation from that expectation, therefore, would be a rather clean probe of the basic four-fermion interaction.

Finally one can ask how these results are modified in case the basic interaction is not "V-A" as we have assumed here. For a general four-fermion interaction, the radiative corrections then become dependent on a cut-off [5, 6, 8]. It is known, however, that their numerical importance is rather independent of the detailed form of the interaction. It is likely that this result applies also to the new terms calculated in this paper and that the deviation of P_ℓ from -1 will remain small.

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* Existing experiments on P_ℓ, of course, are still far from that precision.