Chapter 1

Introduction

1.1 Unpolarized Muon Decay Spectrum

The usual, unpolarized decay $\mu^+ \rightarrow e^+\nu_e\overline{\nu}_\mu$ is well-known. While the energies involved (the muon mass is M_μ =105.66 MeV) are typically higher than those in nuclear beta decay, they are still far below the mass of the W^+ (M_W = 81,000 MeV). Therefore, Fermi's theory of beta decay with its four-point interaction view of the weak force, together with the V-A rule, gives an excellent approximation to the spectrum, although, being non-renormalizable, it cannot be strictly correct. Calculations based on this approximation are easily found elsewhere and will not be reproduced here. One excellent discussion of muon decay can be found in the work by E. D. Commins and P. H. Bucksbaum.[1]

The differential transition probability for muon decay is the product of three basic terms. The first term is the total transition rate, $-(A/16) \cdot (M_\mu^5/192\pi^3) \cdot (1 + 4\eta m_e/M_\mu)$, which contains the usual fifth-power dependence on the decay energy (Sargent rule). The constant $A/16$ is extracted from the squared transition-matrix element and, aside from correction factors of order unity, equals $G_F^2/2$, where G_F is the Fermi coupling constant. The quantity $(1+4\eta m_e/M_\mu)$ appears here only to maintain the normalization and does not

[1] E. D. Commins and P. H. Bucksbaum, *Weak Interactions of Leptons and Quarks* (Cambridge University Press, Cambridge, 1983) pp. 93-108.

deviate far from unity; m_e is the positron mass and η is discussed below. The second term is an x^2 factor due to the positron's available phase space. Here, $x = E_e/E_e(max)$, with E_e being the positron energy and $E_e(max) = \frac{M_\mu^2+m_e^2}{2M_\mu}$ being the spectrum endpoint energy. At $x = 1$, the massless particles are ejected in one direction and the positron in the opposite one, with about half of the energy being carried off in each direction. The third term contains the x-dependence of the squared transition-matrix element, integrated over the momenta of the neutrinos, which escape undetected. This term contains two parameters, ρ and η, which are determined by the spin(s) and coupling(s) of the exchanged boson(s). A more complete discussion appears in Chapter 2.

The differential transition probability, neglecting m_e in most terms for clarity, is

$$\frac{d\Gamma^{(0)}(x)}{dx} = -\frac{A}{16}\frac{M_\mu^5}{192\pi^3}x^2\left[12(1-x)+\frac{8}{3}\rho(4x-3)+24\eta\frac{m_e}{M_\mu}\frac{1-x}{x}\right] . \qquad (1.1)$$

Figure 1 shows the x-dependence of this expression for $\rho \equiv \frac{3}{4}$ and $\eta \equiv 0$, which is predicted theoretically by the two-component neutrino hypothesis; the total transition-rate factor has been dropped to display the energy spectrum, normalized to 1.

That η only appears in Equation 1.1 multiplied by $\frac{m_e}{M_\mu} \approx 0.00967$ means that the spectrum shape is dominated by ρ. However, because of the x^{-1} factor, η becomes increasingly important at low energies. This is shown in Figure 2, which plots the fractional effect of η on the spectrum shape. Note that the divergence, predicted by Equation 1.1 at $x = 0$, does not appear due to radiative corrections. These will be discussed in Sections 2.4 and 2.5.

The relative insensitivity of the unpolarized spectrum shape to η is reflected in the previously extracted experimental values:

$$\rho = 0.7518 \pm 0.0026^2 \quad \text{and} \quad \eta = -0.12 \pm 0.21^3 .$$

While these values are consistent with the predictions of the two-component neutrino theory, the accuracy of η from this direct measurement is markedly less than that of ρ.

[2] G. P. Yost et al., Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).

[3] S. E. Derenzo, Phys. Rev. **181**, 1854 (1969).

1.2 Spectrum Measurement Procedure

In concept, the measurement of the η parameter is very straightforward. Muons (μ^+ to avoid atomic or nuclear capture) are stopped in a thin target, where they decay; a narrow line-width spectrometer is used to measure the momentum spectrum intensity at several points in each of three regimes:

- The kinematic endpoint provides a momentum calibration for the spectrometer at $P_e = P_e(max) = \frac{M_\mu^2 - m_e^2}{2M_\mu}$. Also, because of the extremely sharp drop in intensity at the endpoint, as well as the fact that the spectrum shape here is well-known and almost independent of η, it is possible to verify the calculated line shape of the spectrometer: one simply folds the calculated line shape with the theoretical spectrum and compares to the measurement in this region.

- The midrange of the spectrum is also fairly insensitive to η and, therefore, when data are fit to the theory spectrum with two free parameters, η and amplitude, provides amplitude normalization. This allows the lower-energy data to influence mostly the determination η. While the event rate in the midrange is not as high as that nearer the endpoint, the beam time required for adequate normalization is not excessive, and the non-statistical uncertainties are smaller. One of these is the effect of ρ, which largely determines the shape of the upper half of the unpolarized spectrum; while the amplitude of the lower half is also sensitive to ρ, its *shape* is much less so. This will be discussed further in Section 2.7.

- The region between $x = 0.1$ and $x = 0.4$ holds the maximum statistical sensitivity to η for a narrow line-width spectrometer. This is shown in Figure 3 in terms of the beam time required for the spectrometer used in this experiment to achieve a statistical precision in η of ± 0.088, for a fixed amplitude normalization and a muon flux of 40K/s. One point to be made is that the lower statistical sensitivity for $x < 0.1$ is partly the result of the spectrometer type. If P_{tune} is the mean of the momentum acceptance at a given magnetic field, and ΔP_{accept} is the FWHM of that acceptance, then $\Delta P_{accept} \propto P_{peak} \approx E_{peak}$). Thus, the data rate for small x varies nearly as x^3, rather than as x^2 like the spectrum intensity.

An important thing to note about the procedure as outlined above is that the experiment is self-normalizing. The analysis is complicated by various energy-dependent corrections (for scattering, detection efficiency and so forth), but the experiment is fundamentally simple.

1.3 Muon Beam Characteristics

In an experiment such as this, one needs both very large numbers of μ^+ (to allow adequate statistics, despite the few accepted e^+ at any given, η-sensitive P_{tune}) and a beam with a minimum of range straggling (so that the thickness of the stopping target and, consequently, spectrum distortions can be minimized). The FWHM range straggling, ΔR, for muons in the momentum range of interest, has been approximated as[4]

$$\Delta R = \sqrt{(0.10)^2 + \left(\frac{7}{2}\frac{\Delta P}{P}\right)^2}\, R\ , \tag{1.2}$$

where

$$R \propto P^{7/2}\ .$$

P is the muon momentum and ΔP is the FWHM momentum spread of the beam; it is assumed that $\Delta P/P \ll 1$. The first term of Equation 1.2 then gives the intrinsic straggling, while the second term is due to the momentum spread of the beam; the contributions are equal in this formula when $\Delta P/P = 2.9\%$.

Since the absolute range straggling will decrease for muon beams with low momentum and small spread in momenta, achieving small ΔR requires a low-momentum beam with enough intensity that adequate rate resides in a small momentum slice. Thus, it is the meson factories such as TRIUMF that made this experiment practical with their "surface muon beams," produced by pions decaying at rest near the production target surface.

[4] A. E. Pifer, T. Bowen and K. R. Kendall, Nucl. Instrum. Methods **135**, 39 (1976).

These beams have $P_\mu \approx \frac{M_\pi^2 - M_\mu^2}{2M_\pi} = 29.80$ MeV/c and, for a momentum bite of 2% in the M13 beam line at TRIUMF, muon rates in excess of 10^5/s.

In actuality, the intrinsic straggling is not as high as that given by Equation 1.2, at least in low-Z materials. It is also more complicated, with scattering giving rise to both material and geometrical dependence in stopping distributions, as well as a long tail toward short ranges. The distribution is best determined by Monte Carlo calculation, as discussed in Section 6.1.1.

1.4 Studies with a Positron Beam

In addition to μ^+, the M13 beam line at TRIUMF delivers several other particle species. None of these presented a significant difficulty in this experiment, and the e^+ content was of some value for various checks and calibrations.

One important calibration is that of the target counter scintillator in terms of energy deposition by e^+'s. By comparing the counter's output from accepted decay e^+ with that from e^+ of known energy passing through a known thickness, it is possible to measure the thickness through which the accepted decay e^+ pass.

Another procedure performed was the search for contamination of the low-energy portion of the spectrum by high-energy e^+. For this, the spectrometer was positioned with the beam e^+'s entering the angular acceptance; data were taken at various settings of the magnetic field. While it is not practical to use this information directly to correct the spectrum, it provides a useful check on the existence of contaminations and their approximate size.

Finally, the e^+ data provided a check on the spectrometer momentum calibration. The beam-line tune at the surface muon momentum is known, and positrons of this momentum define a secondary calibration point.

1.5 Systematic Effects

The systematic errors are displayed in Table 6.8 on page 92, and a discussion of the error estimates appears in Appendix E. As shown in the table, the primary systematic effects derive from decay positron interactions in the spectrometer: in the plastic scintillator used to stop the muons, in the counters which detect the decay positrons and in the intervening material of the spectrometer. Measurements of the low-energy portion of the spectrum are very vulnerable to contamination by positrons of higher energy which lose energy by various processes.

Energy losses in the muon-stop target, including the large ones due to Bhabha scattering and bremsstrahlung, make a major contribution. While correction for this can be made in principle, uncertainties in the muon depth in the target, in the relevant cross-sections, in counter calibrations and in the Monte Carlo calculations leave some error.

The detection efficiency for positrons is energy dependent due to scattering and annihilation, which is not completely corrected due to uncertainties in counter calibrations and geometry, as well as approximations in its calculation. The major interactions in other parts of the spectrometer appear in Table 6.8 under the entries for "back plate," "cables," "$K2$" and "$C1$." The relevant geometry is shown in Figure 8. Estimation of these processes is affected by several things, including approximations and statistics in Monte Carlos, counter calibrations, geometrical errors and uncertainties in the cross sections for the physical processes. The "back plate" correction, made for contamination due to showering in the spectrometer back plate, has an empirically-determined component in addition to the Monte Carlo result, and is limited in accuracy mostly by statistics on the measurement.

The other significant error, listed in Table 6.8 as "μ^+ spin angle," reflects one mechanism for the muon polarization not being completely canceled in the measurement. This would occur for the partially-polarized muons of this experiment if their spins were not, on the average, perpendicular to the spectrometer axis.

Chapter 2

The Muon Decay Spectrum

2.1 Spectrum for General Interactions

Within the framework of the Weinberg-Salaam theory, muon decay has been calculated[1,2] to second order. The conclusion is that, for reasonable masses of the Higgs boson, the spectrum shape differs only negligibly from that found using the four-point Fermi interaction, provided that the Fermi coupling constant is suitably redefined:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}[1 + O(\alpha)] \, .$$

There is then no difficulty in retaining the four-point interaction formalism, and we shall do so.

There are several conventions which are currently used in writing a general Hamiltonian for muon decay. For historical reasons relating to the non-detection of decay neutrinos, the most common form is the charge-retention ordering, and it is for this form that results are

[1] D. A. Ross, Nucl. Phys. **B51**, 116 (1973).

[2] A. Donnachie and J. Mohammad, CERN Report TH-2132 (1976).

usually specified:

$$\mathcal{H}_{cr} = \sum_j \left(\overline{\Psi}_e O_j \Psi_\mu\right) \overline{\Psi}_{\nu_\mu} O_j (C_j + C'_j \gamma_5) \Psi_{\nu_e} + h.c.$$

$$j = S, V, T, A, P$$

$$O_S = 1,\ O_V = \gamma^\mu,\ O_T = \frac{1}{\sqrt{2}} \sigma^{\mu\nu},\ O_A = \gamma^\mu \gamma_5,\ O_P = i\gamma_5\ .$$

However, another form often appears in theoretical calculations; this charge-exchange order is more physical, given the success of the Weinberg-Salaam theory:

$$\mathcal{H}_{ce} = \sum_j \left(\overline{\Psi}_{\nu_\mu} O_j \Psi_\mu\right) \overline{\Psi}_e O_j (\hat{C}_j + \hat{C}'_j \gamma_5) \Psi_{\nu_e} + h.c.$$

The O_j are defined as before.

In either of these forms there are, most generally, 10 complex constants, C_j,C'_j or $\hat{C}_j$,$\hat{C}'_j$. Dismissing one variable for the arbitrary, overall phase, 19 real parameters remain, in general, to be determined by experiment. Since $\mathcal{H}_{cr}$ and $\mathcal{H}_{cc}$ must be physically equivalent, it is clear that the parameters of the two representations are closely related. Conversions between C_j and $\hat{C}_j$, or between C'_j and $\hat{C}'_j$, are given by linear Fierz[3] transformations:

$$C_i = \sum_j \Lambda_{ij} \hat{C}_j\ , \quad \hat{C}_i = \sum_j \Lambda_{ij} C_j\ ,$$

$$C'_i = \sum_j \Lambda_{ij} \hat{C}'_j\ , \quad \hat{C}'_i = \sum_j \Lambda_{ij} C'_j\ .$$

The explicit form of (Λ_{ij}) is

$$(\Lambda_{ij}) = \frac{1}{4} \begin{pmatrix} -1 & -4 & -6 & 4 & 1 \\ -1 & 2 & 0 & 2 & -1 \\ -1 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & -4 & 6 & 4 & -1 \end{pmatrix} .$$

Naturally, (Λ_{ij}) would be different for another definition of the O_j operators; the lack of a standard definition amongst authors (and their frequent failure to specify the convention used) makes the literature a hazardous place in which to compare theoretical discussions.

[3] M. Fierz, Z. Phys. **104**, 553 (1937).

There is also another notation which has been developed by Scheck.[4,5] This is the helicity-projection form in which terms correspond to states of definite helicity for massless particles, simplifying expressions in many treatments. This form will not be used in this thesis and is mentioned only for completeness.

Independent of the convention used, the $\mu^{\pm}$ decay spectrum without radiative corrections is given below for the case when all aspects of the $e^{\pm}$ are measured and neither of the neutrinos is detected.[6,7] Note that only 10 real parameters involving the interaction coupling constants appear in the spectrum formula, meaning that nine parameters are undetermined in the most general case, when neutrinos are not observed:

$$\begin{aligned}\frac{d^2\Gamma^{(0)}(x,\theta,\phi,\psi)}{dx\,d(\cos\theta)} = {} & 6\frac{A}{16}\frac{M_\mu^5}{192\pi^3}\left(1+\frac{m_e^2}{M_\mu^2}\right)^4\sqrt{x^2-x_0^2}\cdot \\ & \Big\{\left[x(1-x)+\tfrac{2}{9}\rho(4x^2-3x-x_0^2)+\eta x_0(1-x)\right] \\ & \pm \tfrac{1}{3}\xi\sqrt{x^2-x_0^2}\cos\theta\left[1-x+\tfrac{2}{3}\delta\left(4x-3-\tfrac{m_e}{M_\mu}x_0\right)\right] \\ & \pm \xi'\sqrt{x^2-x_0^2}\cos\phi\left[1-x+\tfrac{2}{3}\delta'\left(4x-3-\tfrac{m_e}{M_\mu}x_0\right)\right] \\ & + \tfrac{1}{3}\xi''\cos\theta\cos\phi\left[x(1-x)+\tfrac{2}{3}\rho'(4x^2-3x-x_0^2)+\eta' x_0(1-x)\right] \\ & + \sin\theta\sin\phi\cos\psi\left[(1-x)x_0\tfrac{3a-2b-2c}{3A}+x(1-x)\tfrac{\alpha}{A}+(x-x_0^2)\tfrac{2\beta}{3A}\right] \\ & + \sin\theta\sin\phi\sin\psi\sqrt{x^2-x_0^2}\left[(1-x)\tfrac{\alpha'}{A}+\tfrac{2}{3}\left(1-\tfrac{m_e}{M_\mu}x_0\right)\tfrac{\beta'}{A}\right]\Big\}\ .\end{aligned}$$

As before, $x = E_e/E_e(max)$, where $E_e(max) = (M_\mu^2 + m_e^2)/2M_\mu$; $x_0 = m_e/E_e(max)$. Obviously, one has

$$x_0 \le x \le 1\ .$$

In order to specify the angles, $\vec{\zeta_\mu}$ is defined as the direction of the muon spin, $\vec{P_e}$ as the momentum of the emitted $e^{\pm}$ and $\vec{\zeta_e}$ as its spin direction. Then, θ is the angle between $\vec{P_e}$ and $\vec{\zeta_\mu}$; ϕ is the angle between $\vec{P_e}$ and $\vec{\zeta_e}$; and ψ is the azimuthal angle by which $\vec{\zeta_e}$ is rotated from $\vec{\zeta_\mu}$, around $\vec{P_e}$.

[4] F. Scheck, in *Leptons, Hadrons and Nuclei* (North Holland, Amsterdam, 1983), Chap. 5, sec. 6.2.2.

[5] K. Mursula and F. Scheck, Nucl. Phys. **B253**, 189 (1985).

[6] T. Kinoshita and A. Sirlin, Phys. Rev. **108**, 844 (1957).

[7] F. Scheck, Phys. Reports **44**, 187 (1978).

The remaining variables depend only upon the coupling constants and are given here for the charge-retention form; the following real, bilinear combinations are defined as

$$
\begin{aligned}
a &= |C_S|^2 + |C_S'|^2 + |C_P|^2 + |C_P'|^2 \\
\alpha &= |C_S|^2 + |C_S'|^2 - |C_P|^2 - |C_P'|^2 \\
b &= |C_V|^2 + |C_V'|^2 + |C_A|^2 + |C_A'|^2 \\
\beta &= |C_V|^2 + |C_V'|^2 - |C_A|^2 - |C_A'|^2 \\
c &= |C_T|^2 + |C_T'|^2 \\
a' &= 2Re(C_S C_P'^* + C_S' C_P^*) \\
b' &= 2Re(C_V C_A'^* + C_V' C_A^*) \\
c' &= -2Re(C_T C_T'^*) \\
\alpha' &= 2Im(C_S C_P'^* + C_S' C_P^*) \\
\beta' &= 2Im(C_V C_A'^* + C_V' C_A^*)
\end{aligned}
$$

or, using the Fierz transformation, as

$$
\begin{aligned}
a &= 2\left(|\hat{C}_V - \hat{C}_A|^2 + |\hat{C}_V' - \hat{C}_A'|^2\right) + \tfrac{1}{8}\left(|\hat{C}_S + 6\hat{C}_T - \hat{C}_P|^2 + |\hat{C}_S' + 6\hat{C}_T' - \hat{C}_P'|^2\right) \\
\alpha &= Re\left[(\hat{C}_V - \hat{C}_A)(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)^* + (\hat{C}_V' - \hat{C}_A')(\hat{C}_S' + 6\hat{C}_T' - \hat{C}_P')^*\right] \\
b &= \tfrac{1}{2}\left(|\hat{C}_V + \hat{C}_A|^2 + |\hat{C}_V' + \hat{C}_A'|^2\right) + \tfrac{1}{8}\left(|\hat{C}_S + \hat{C}_P|^2 + |\hat{C}_S' + \hat{C}_P'|^2\right) \\
\beta &= -\tfrac{1}{2}Re\left[(\hat{C}_V + \hat{C}_A)(\hat{C}_S + \hat{C}_P)^* + (\hat{C}_V' + \hat{C}_A')(\hat{C}_S' + \hat{C}_P')^*\right] \\
c &= \tfrac{1}{4}\left(|\hat{C}_S - 2\hat{C}_T - \hat{C}_P|^2 + |\hat{C}_S' - 2\hat{C}_T' - \hat{C}_P'|^2\right) \\
a' &= Re\left[4(\hat{C}_V - \hat{C}_A)(\hat{C}_V' - \hat{C}_A')^* - \tfrac{1}{4}(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)(\hat{C}_S' + 6\hat{C}_T' - \hat{C}_P')^*\right] \\
b' &= Re\left[(\hat{C}_V + \hat{C}_A)(\hat{C}_V' + \hat{C}_A')^* - \tfrac{1}{4}(\hat{C}_S + \hat{C}_P)(\hat{C}_S' + \hat{C}_P')^*\right] \\
c' &= -\tfrac{1}{8}Re\left[(\hat{C}_S - 2\hat{C}_T - \hat{C}_P)(\hat{C}_S' - 2\hat{C}_T' - \hat{C}_P')^*\right] \\
\alpha' &= -Im\left[(\hat{C}_V - \hat{C}_A)(\hat{C}_S' + 6\hat{C}_T' - \hat{C}_P')^* + (\hat{C}_V' - \hat{C}_A')(\hat{C}_S + 6\hat{C}_T - \hat{C}_P)^*\right] \\
\beta' &= \tfrac{1}{2}Im\left[(\hat{C}_V + \hat{C}_A)(\hat{C}_S' + \hat{C}_P')^* + (\hat{C}_V' + \hat{C}_A')(\hat{C}_S + \hat{C}_P)^*\right] .
\end{aligned}
$$

From these, the muon decay parameters are defined as

$$
\begin{aligned}
A &= a + 4b + 6c \\
\rho &= \tfrac{1}{A}(3b + 6c) \\
\eta &= \tfrac{1}{A}(\alpha - 2\beta) \\
\xi &= -\tfrac{1}{A}(3a' + 4b' - 14c') \\
\delta &= \tfrac{1}{A\xi}(-3b' + 6c') \\
\xi' &= -\tfrac{1}{A}(a' + 4b' + 6c') \\
\delta' &= -\tfrac{1}{A\xi'}(b' + 2c') \\
\xi'' &= \tfrac{1}{A}(3a + 4b - 14c) \\
\rho' &= \tfrac{1}{A\xi''}(3b - 6c) \\
\eta' &= \tfrac{1}{A\xi''}(3\alpha + 2\beta) \;.
\end{aligned}
$$

Substitution into the above expressions gives the explicit definition of η; in the charge-retention form it is

$$\eta = \frac{1}{A}\left[|C_S|^2 + |C_S'|^2 - 2\left(|C_V|^2 + |C_V'|^2 - |C_A|^2 - |C_A'|^2\right) - |C_P|^2 - |C_P'|^2\right] \;,$$

where

$$A = |C_S|^2 + 4|C_V|^2 + 6|C_T|^2 + 4|C_A|^2 + |C_P|^2 \;+\; (\text{primed terms}) \;;$$

and, in the charge-exchange form, it is

$$\eta = \frac{2}{A} Re\left[\hat{C}_V\hat{C}_S^* + \hat{C}_V'\hat{C}_S'^* + \hat{C}_A\hat{C}_P^* + \hat{C}_A'\hat{C}_P'^* + 3(\hat{C}_V - \hat{C}_A)\hat{C}_T^* + 3(\hat{C}_V' - \hat{C}_A')\hat{C}_T'^*\right] \;,$$

where

$$A = |\hat{C}_S|^2 + 4|\hat{C}_V|^2 + 6|\hat{C}_T|^2 + 4|\hat{C}_A|^2 + |\hat{C}_P|^2 \;+\; (\text{primed terms}) \;.$$

2.2 Physical Motivations

There is presently no compelling evidence of inconsistency with a universal V-A interaction given by

$$(\hat{C}_j) = (C_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad (\hat{C}'_j) = (C'_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

There is also no evidence of neutral decay products aside from a massless (or near massless) neutrino, anti-neutrino and photons. Other possibilities are not excluded, however, and some of these will be discussed below.

2.2.1 Limitations of Positron-Inclusive Measurements

It was previously noted that only 10 of the 19 parameters allowed in the most general Hamiltonian can be measured in muon decay, if neutrinos are not detected. One can, however, ask how much ambiguity would remain in an ideal world of completely accurate measurements on the $e^{\pm}$, if the commonly accepted V-A description is correct. The answer is that only two, rather than nine, degrees of freedom would remain. Looking at the charge-retention formalism on page 10, one can see that this is because a showing that $a = c = a' = c' = \alpha' = 0$ eliminates 12 degrees of freedom. The two remaining degrees of freedom are illustrated below by expressions for $(\hat{C})$ and $(\hat{C}')$; the positron spectrum is completely independent of the value of ϵ, which is not necessarily either small or real:

$$(\hat{C}_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} \epsilon \\ 1 \\ 0 \\ 1 \\ \epsilon \end{pmatrix}, \qquad (\hat{C}'_j) = \frac{G_F}{\sqrt{2}} \begin{pmatrix} \epsilon \\ -1 \\ 0 \\ -1 \\ \epsilon \end{pmatrix}.$$

The Hamiltonian corresponding to this would be

$$\begin{aligned}\mathcal{H}_{ce} &= \tfrac{G_F}{\sqrt{2}}\overline{\Psi}_{\nu_\mu}\gamma^\nu(1-\gamma_5)\Psi_\mu\overline{\Psi}_e\gamma_\nu(1-\gamma_5)\Psi_{\nu_e} \\ &+ \epsilon\tfrac{G_F}{\sqrt{2}}\overline{\Psi}_{\nu_\mu}(1-\gamma_5)\Psi_\mu\overline{\Psi}_e(1+\gamma_5)\Psi_{\nu_e} + h.c.\ ,\end{aligned}$$

which shows that this remaining ambiguity would be eliminated by an helicity measurement upon either of the neutrinos; no such measurement has yet been made, and it is upon faith that "ν_e" and "ν_μ" are taken to be the same particles as those found in nuclear beta decay and π^+ decay, respectively.

A possible source of this type of Hamiltonian is the exchange of a charged Higgs boson ϕ'', in addition to the usual W_L^+, when there are (possibly) massive neutrinos of opposite helicity to the usual ones. A paper by Fayet[8] develops this possibility. The *effective* Hamiltonian for the decay positrons would be as above, with

$$\epsilon = \frac{m_{\nu'_\mu} m_{\nu'_e}}{m^2_{\phi''_\pm}}\ .$$

Fayet claims that the positron spectrum would be altered in this case (specifically, that ρ would deviate from 3/4), though this is contradicted by the above discussion. One could obtain Fayet's result if one were to calculate $\hat{C}$,$\hat{C}'$ and then inadvertently use them in formulae intended for C,C'.

2.2.2 Lorentz-Structure Physics

Next we will consider what physics might be found as measurements of the $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$ parameters are improved. The following discussion draws heavily upon a paper by Mursula, Roos and Scheck.[9] The notation, however, has been changed for consistency here.

If one assumes that muon decay is mediated by heavy, charged bosons with spins of 0, 1 and/or 2, the effective four-fermion interaction becomes

$$\mathcal{H}_{ce} = \frac{G_F}{\sqrt{2}}\left[\sum_i K^{(i)\dagger}K^{(i)} + \sum_i J_\alpha^{(i)\dagger}J^{(i)\alpha} + \sum_i T_{\alpha\beta}^{(i)\dagger}T^{(i)\alpha\beta}\right]\ ,$$

[8] P. Fayet, Nucl. Phys. **B78**, 14 (1974).

[9] K. Mursula, M. Roos and F. Scheck, Nucl. Phys. **B219**, 321 (1983).

where the index i refers to the different charged bosons of a given spin and where

$$\begin{aligned}
K^{(i)} &= g^e_{S,i}(\bar{e}1\nu_e) + g^\mu_{S,i}(\bar{\mu}1\nu_e) + g^e_{P,i}(\bar{e}\gamma_5\nu_e) + g^\mu_{P,i}(\bar{\mu}\gamma_5\nu_e) \ , \\
J^{(i)}_\alpha &= g^e_{V,i}(\bar{e}\gamma_\alpha\nu_e) + g^\mu_{V,i}(\bar{\mu}\gamma_\alpha\nu_\mu) + g^e_{A,i}(\bar{e}\gamma_\alpha\gamma_5\nu_e) + g^\mu_{A,i}(\bar{\mu}\gamma_\alpha\gamma_5\nu_\mu) \ , \\
T^{(i)}_{\alpha\beta} &= g^e_{T,i}(\bar{e}\sigma_{\alpha\beta}\nu_e) + g^\mu_{T,i}(\bar{\mu}\sigma_{\alpha\beta}\nu_\mu) + g^e_{T',i}(\bar{e}\sigma_{\alpha\beta}\gamma_5\nu_e) + g^\mu_{T',i}(\bar{\mu}\sigma_{\alpha\beta}\gamma_5\nu_\mu) \ .
\end{aligned}$$

The constants $g^e_{l,i}$ and $g^\mu_{l,i}$ are the coupling constants for the heavy boson of mass m_i, except for a factor $2^{1/4}/m_i\sqrt{G_F}$. These constants are related to our previous notation by

$$\begin{aligned}
\hat{C}_S &= \textstyle\sum_i g^e_{S,i}g^{\mu *}_{S,i} \\
\hat{C}_V &= \textstyle\sum_i g^e_{V,i}g^{\mu *}_{V,i} \\
\hat{C}_T &= \textstyle\sum_i \left(g^e_{T,i}g^{\mu *}_{T,i} - g^e_{T',i}g^{\mu *}_{T',i}\right) \\
\hat{C}_A &= \textstyle\sum_i g^e_{A,i}g^{\mu *}_{A,i} \\
\hat{C}_P &= \textstyle\sum_i g^e_{P,i}g^{\mu *}_{P,i} \\
\hat{C}'_S &= \textstyle\sum_i g^e_{P,i}g^{\mu *}_{S,i} \\
\hat{C}'_V &= \textstyle\sum_i g^e_{A,i}g^{\mu *}_{V,i} \\
\hat{C}'_T &= \textstyle\sum_i \left(g^e_{T,i}g^{\mu *}_{T',i} - g^e_{T',i}g^{\mu *}_{T,i}\right) \\
\hat{C}'_A &= \textstyle\sum_i g^e_{V,i}g^{\mu *}_{A,i} \\
\hat{C}'_P &= \textstyle\sum_i g^e_{S,i}g^{\mu *}_{P,i} \ .
\end{aligned}$$

These formulae can be seen to predict relationships between the $\hat{C}_l$ constants, under rather weak assumptions: if there is no more than one exchanged boson with zero spin, one obtains

$$\hat{C}'_S\hat{C}'_P = \hat{C}_S\hat{C}_P$$

or, if there is no more than one exchanged boson with unit spin, one obtains

$$\hat{C}'_V\hat{C}'_A = \hat{C}_V\hat{C}_A \ .$$

There are further relationships in the event that lepton universality ($g^e_{l,i} = g^\mu_{l,i}$) or weak universality ($g^e_{l,i} = (M_\mu/m_e)g^\mu_{l,i}$) holds.

(Pseudo)scalar Exchange

One class of theories affecting η contains scalar or pseudoscalar exchange in addition to the usual V-A structure. If strict lepton universality holds, the expression for η becomes

$$\eta = -2\frac{\beta}{A} = \frac{2|g_S|^2 + 2|g_P|^2 - 4Re(g_S g_P^*)}{16|g_V|^2 + (|g_S|^2 + 2|g_P|^2)^2} .$$

However, a significant pseudoscalar coupling is almost excluded by existing measurements[10] of the $\Gamma(\pi^+ \to e^+\nu_e)/\Gamma(\pi^+ \to \mu^+\nu_\mu)$ reaction. While it can be argued that this reaction is not purely leptonic, and that this causes some theoretical uncertainty in its interpretation, the more intriguing case evades these limits through weak universality. This would be the case for a charged Higgs boson, which exists in some non-minimal models. Thus, η becomes

$$\eta = -2\frac{\beta}{A} = \frac{M_\mu}{m_e}\frac{2|g_S^e|^2 + 2|g_P^e|^2 - 4Re(g_S^e g_P^{e*})}{16|g_V^e|^4 + \frac{M_\mu^2}{m_e^2}(|g_S^e|^2 + 2|g_P^e|^2)^2} .$$

These physics are slightly clarified through yet another change of notation, as used by Mursula.[11] If we define

$$g_i^e = \frac{M_W}{M_H}\frac{M_e}{M}c_i ,$$

where M and c_i $(i = S, P)$ are an unknown mass scale and coupling constants, respectively, we can further define

$$\lambda_i = \frac{M_\mu}{m_e}|\frac{g_i^e}{g_V}| = \frac{m_e m_\mu}{M^2}\frac{M_W^2}{M_H^2}|\frac{c_i}{g_v}|^2 , \quad \alpha_s = arg(c_S c_P^*) .$$

Expressions for η and $1 - \xi$ then become

$$\eta = \frac{\lambda_S + \lambda_P - 2\sqrt{\lambda_S \lambda_P}\cos\alpha_s}{16 + (\lambda_S + \lambda_P)^2}$$

and

$$1 - \xi = (\lambda_S + \lambda_P)\eta .$$

Clearly, the sensitivity of these parameters to the hypothetical Higgs physics depends strongly upon the unknown mass scale M. *If*, Mursula points out, $c_i \approx g_v$, $M \approx M_H$ and

[10] D. Bryman *et al.*, Phys. Rev. Lett. **50**, 7 (1983).

[11] K. Mursula, Univ. Bern Report BUTP-84/25 (1984).

$M_H > 20$ GeV, it would be doubtful that one could ever observe the effect on muon decay, whether or not it existed. Also, the sensitivity would vanish for $\alpha_S = 0$ and $\lambda_S = \lambda_P$. Nonetheless, the possibility of observing these physics in muon decay should not be rejected.

Spin-2 Exchange

In the most general case, inspection of the formula for η shows that there is possible sensitivity to tensor currents. However, $\hat{C}_T^*$,$\hat{C}_T'^*$ appear with the factors $\hat{C}_V - \hat{C}_A$ and $\hat{C}_V' - \hat{C}_A'$, respectively, so the sensitivity is reduced or eliminated, due to the small upper limit on these factors.

Alternatively, it might be said that η is sensitive to right-left (broken) symmetries only in the presence of fairly strong tensor currents. Thus, at least two hypothetical extensions to the standard model must exist for measurements of η to detect either one. This makes any search for these possibilities, using η, very speculative.

2.2.3 Massive Mixed Neutrinos

It has been pointed out[12,13,14] that the weak-interaction eigenstates of neutrinos are not necessarily mass eigenstates, so that many of the conventional mass limits on neutrinos do not exclude the possibility that ν_e and/or ν_μ might contain a small admixture of a heavy neutrino. Such an admixture would manifest itself by the incoherent addition of one or more muon decay spectra to the main one with endpoints at

$$(x_{max})_{i,j} = 1 - \frac{(m_{\nu_i} + m_{\nu_j})^2}{M_\mu^2 + m_e^2} ,$$

where m_{ν_i} and m_{ν_j} are the masses of the emitted neutrinos. This being so, one should consider the possible effects of massive neutrino mixing in an analysis of the muon decay spectrum; the Lorentz structure cannot be assumed to be the sole determining factor.

[12] A. Sirlin, Proceedings of the TRIUMF Muon Physics/Facility Workshop, p.81 (1980).

[13] R. E. Shrock, Phys. Rev. D **24**, 1275 (1981).

[14] P. Kalyniak and J. N. Ng, Phys. Rev. D **24**, 1874 (1981).

In particular, massive neutrinos could result in a measurement of $\eta \neq 0$, even if weak interactions were purely $V - A$. In the paper by Shrock referenced above, Fig. 25 graphs an effective value of η when there is a massive component to one the neutrinos and ρ,η are allowed to vary freely to provide the best fit to the unpolarized muon decay spectrum. However, one must be careful in interpreting this figure, because it has been calculated for one specific experiment. Even then, it is only accurate if one attempts to extract the massive neutrino information from a value of η_{eff}; this is not the best approach because the effect of massive, mixed neutrinos on the spectrum has an entirely different energy dependence than does a truly non-zero value of η.

To make this explicit, consider a muon decay with one neutrino of mass m_{ν_i}. We define $d = (m_{\nu_i}/M_\mu)^2$, drop terms in m_e and retain our usual definition of x. Also, $\Theta(s) \equiv 1$ for $s \geq 0$ and $\Theta(s) \equiv 0$ for $s < 0$. Then, the spectrum supplement is proportional to

$$2x^2\left(1 - \frac{d}{1-x}\right)^2\left(3 - 2x + d\frac{3-x}{1-x}\right)\Theta(x_{max} - x) \ ,$$

which for $x \ll 1$ becomes

$$6x^2(1-d)^2(1+d)\,\Theta(x_{max} - x) \ .$$

This clearly has little in common with the usual term proportional to η:

$$12\,\eta\, x_0\, x(1-x) \ .$$

An additional problem exists when considering the effect upon the low-energy part of the decay spectrum: the radiative corrections for massive neutrinos differ from those for the massless case; the correction terms are of order $(m_{\nu_i}/M_\mu)^2$ and higher, where ν_i is the massive neutrino component. A calculation has been performed[15] in which terms proportional to powers of m_e were dropped, but which allows for massive neutrinos. The main effect is to reduce the spectrum at low energies and near the endpoint. This is just as one would expect: the effect on the bulk of the spectrum is small since the neutrinos are

[15] P. Kalyniak and J. N. Ng, Phys. Rev. D **25**, 1305 (1982).

not involved in the photon diagrams directly. However, the spectrum near the endpoint is reduced by the emission of soft photons and, since fewer high-energy positrons are kinematically allowed, hard photon emission does not increase the low-energy spectrum as much as in the massless neutrino case.

In general, pseudoscalar decays are intrinsically more sensitive to massive neutrinos than the 3-body decay of muons: the decay $\pi^+ \to \mu^+ \nu_\mu$ has been used to establish[16] $|U_{\mu i}|^2 < 2 \times 10^{-4}$ for 10 MeV $< m_{\nu_i} <$ 30 MeV, and the decay $K^+ \to \mu^+ \nu_\mu$ has been used to set[17] the limit of $|U_{\mu i}|^2 < 10^{-4}$ for 70 MeV $< m_{\nu_i} <$ 335 MeV. For m_{ν_i} between 30 MeV and 70 MeV, however, the current limits are relatively weak; Shrock[18] has used the ρ parameter of muon decay to set the limit $|U_{\mu i}|^2 < 10^{-2}$ for 12 MeV $< m_{\nu_i} <$ 63 MeV. This limit is tightest near $m_{\nu_i} = 40$ MeV, where $|U_{\mu i}|^2 < 2 \times 10^{-3}$. Above this range, the best limits come from an analysis[19] of existing data on tritium recoil in the reaction $\mu^- + {}^3He \to \nu_\mu + {}^3H$. The limit set is $|U_{\mu i}|^2 < 10^{-2}$ for 60 MeV $< m_{\nu_i} <$ 72 MeV.

For 30 MeV $< m_{\nu_i} <$ 70 MeV, especially, it would be desirable to improve the limits on $|U_{\mu i}|^2$, although the current limit on the tau neutrino mass ($m_{\nu_\tau} <$ 35 MeV at the 95% confidence level[20]) reduces the motivation. High-statistics muon decay spectrum measurements may improve limits on $|U_{\mu i}|^2$ for m_{ν_i} in, at least, part of this range. In line with comments earlier in this section, it is possible to obtain more accurate limits on $|U_{\mu i}|^2$ than a casual reading of Shrock's paper might lead one to believe.

2.2.4 Light Scalar Neutrinos

As Buchmüller and Scheck[21] have pointed out, if neutrinos and the W have supersymmetric partners ($\tilde{\nu}_e$, $\tilde{\nu}_\mu$ and $\tilde{W}$) and the scalar neutrinos are light enough for the decay to be

[16] R. Abela *et al.*, Phys. Lett. **105B**, 263 (1981).

[17] R. S. Hayano, Phys. Rev. Lett. **49**, 1305 (1982).

[18] R. E. Shrock, Phys. Rev. D **24**, 1275 (1981).

[19] J. P. Deutsch, M. Lebrun and R. Prieels, Phys. Rev. D **27**, 1644 (1983).

[20] G. P. Yost et al., Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).

[21] W. Buchmüller and F. Scheck, Phys. Lett. **145B**, 421 (1984).

kinematically allowed, the positron spectrum from μ^+ decay would be affected. For very light scalar neutrinos, the effect on the spectrum mimics changes in ρ, δ and ξ. In this case, the current values of ρ and δ set similar limits on the mass of the $\tilde{W}$, and the combined limit stated by Buchmüller and Scheck is $M_{\tilde{W}}/M_W > 4$. Polarization measurements are not sensitive to these physics, since the decay positrons are still completely polarized (in the $m_e = 0$ approximation). Also, the combination $\xi\frac{\rho}{\delta}$ is only very weakly sensitive.

The limits from ρ and δ do not apply when the $\tilde{\nu}_e$, $\tilde{\nu}_\mu$ masses are large enough to place the endpoint below the range over which the most accurate ρ and δ measurements were done; one must then look to the lower part of the spectrum. In the limit that $m_e = 0$ and $m_{\tilde{\nu}_e} \ll M_\mu$, the comparison to the usual unpolarized spectrum is given by

$$\frac{d\tilde{\Gamma}/dx}{d\Gamma/dx} = \epsilon\,\Theta(1 - r - x)\left(1 - \frac{r}{1-x}\right)^3 \frac{3-x}{3-2x} \,,$$

where

$$\epsilon = \left(\frac{M_W}{M_{\tilde{W}}}\right)^4, \quad r = \left(\frac{m_{\tilde{\nu}_\mu}}{M_\mu}\right)^2 .$$

As with the massive Dirac neutrinos, this spectrum effect does not mimic a term proportional to η, so a direct fit to the measured spectrum should be used to set limits.

2.2.5 Familons

If the lepton family symmetry is global, its breaking leads to a massless Goldstone boson, G. (By contrast, if the symmetry is local, the breaking leads to mirror fermions.) It has been noted[22] that a measurement of the ratio $\Gamma(\mu \to eG)/\Gamma(\mu \to e\nu\bar{\nu})$ would establish, or place limits on, the breaking scale.

Since the Goldstone boson is massless, the effect for which one is searching is a small spike in the positron energy spectrum at $x = 1$. For this search one would prefer an instrument of moderate acceptance and extremely high resolution, so as to resolve a narrow spike at the endpoint of the conventional spectrum. The Comus spectrometer is, unfortunately,

[22] G. B. Gelmini, S. Nussinov and T. Yanagida, Nucl. Phys. **B219**, 31 (1983).

not well-matched to this search due to its 2.8% FWHM resolution; the width of the spike, by contrast, would be of order $10^{-13}\%$, being spread only by radiative corrections.

2.2.6 Majorons

The spectrum for $\mu \to eMM$, where the Majoron M is a hypothetical Goldstone boson coupling to neutrinos, has been calculated.[23] The same calculation applies to any light scalar or pseudoscalar particle that couples to neutrinos. The comparison to the usual unpolarized spectrum, in the $m_e = 0$ limit, is given by

$$\frac{d\Gamma_M/dx}{d\Gamma/dx} = \frac{24\alpha_M^2}{\pi^2}(\ln\frac{M_W}{M_\mu})^2\frac{1}{3-2x} \approx \frac{107\alpha_M^2}{3-2x} ,$$

where

$$\alpha_M = \sum_i \frac{g^*_{\mu i}g_{ie}}{4\pi}$$

and the g_{ij} are defined by the Majoron-neutrino couplings

$$-ig_{ij}\bar{\nu}_i^c\gamma_5\nu_j M .$$

Clearly, the limits on this model are best provided by measurements of the high-energy part of the spectrum.

Another related effect on the spectrum is Majoron bremsstrahlung, the rate of which was also calculated by Goldman, Kolb and Stephenson in the same work:

$$\begin{aligned}\frac{d\Gamma_B/dx}{d\Gamma/dx} &= \frac{\alpha_B}{8\pi}\left[2\ln\frac{M_\mu}{m_e} - 1 - \ln\frac{x}{2} + \frac{x+\ln(1-x)}{x^2(3-2x)}\right] \\ &\approx \frac{\alpha_B}{8\pi}\left[9.663 - \ln\frac{x}{2} + \frac{x+\ln(1-x)}{x^2(3-2x)}\right] ,\end{aligned}$$

where

$$\alpha_B = \left(|g_{ee}|^2 + |g_{\mu e}|^2 + |g_{\tau e}|^2 + |g_{e\mu}|^2 + |g_{\mu\mu}|^2 + |g_{\tau\mu}|^2\right)/4\pi .$$

While the relative effect on the spectrum would be larger at low energies, better limits would probably be provided by high-energy measurements, where high accuracy can be more easily achieved. The most accurate limit in muon decay would probably not use the

[23] T. Goldman, E. W. Kolb and G. J. Stephenson, Jr., Phys. Rev. D **26**, 2503 (1982).

unpolarized spectrum at all, but, rather, would take advantage of the fact that $d\Gamma_B/dx$ is isotropic; one would derive the limit from measurements near $x = 1$ in the direction opposite to the muon spin (for μ^+) where the rate is almost zero for the conventional decay.

2.2.7 Impact upon ρ Measurements

While the coupling between fitted values of η and ρ does not constitute a physical effect within the meaning of this section, it does provide another motivation to obtain an accurate measurement of η. Derenzo[24] discusses the correlation between the combined measurements of ρ and a measurement of η. His result could be reasonably summarized as

$$\rho = [0.7523 + 0.0044\eta] \pm [0.0024^2 + (0.0044\Delta\eta)^2]^{1/2} .$$

Thus, typical measurements of ρ must either assume that $\eta = 0$ or be limited in their accuracy to about $0.0044\Delta\eta$. This degree of correlation varies, of course, depending upon how a particular measurement of ρ distributes the statistical weight over the spectrum.

2.3 Unpolarized Spectrum

We now specialize the decay rate to a spectrometer in which the acceptance is nearly azimuthally symmetric around an axis at $\theta = 90°$ and the $e^\pm$ spin is not detected. This averages ϕ, the angle between the positron momentum and its spin direction, between 0 and π; ψ, the angle by which the spin is rotated around the momentum, is averaged between 0 and 2π:

$$\frac{d^2\Gamma^{(0)}(x)}{dx\, d(\cos\theta)} = \frac{A}{16}\frac{M_\mu^5}{192\pi^3}\left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 x\sqrt{x^2 - x_0^2}\left[6(1-x) + \frac{4}{3}\rho\left(4x - 3 - \frac{x_0^2}{x}\right) + 6\eta x_0 \frac{1-x}{x}\right] .$$

Integrating over θ, we obtain the unpolarized rate at x:

$$\frac{d\Gamma^{(0)}(x)}{dx} = \frac{A}{16}\frac{M_\mu^5}{192^3}\left(1 + \frac{m_e^2}{M_\mu^2}\right)^4 \mathcal{A}\left(\frac{x}{B}\right) 2x\sqrt{x^2 - x_0^2}\left[6(1-x) + \frac{4}{3}\rho\left(4x - 3 - \frac{x_0^2}{x}\right) + 6\eta x_0 \frac{1-x}{x}\right] .$$

[24] S. E. Derenzo, Phys. Rev. 181, 1854 (1969).

Here we have defined

$$\mathcal{A}(\frac{x}{B}) = \frac{1}{2}\int_0^\pi a(\theta, x/B)\sin\theta\, d\theta ,$$

where B is the magnetic field at some reference point in the spectrometer and $a(\theta, x/B)$ is the probability that a particle emitted at θ with energy corresponding to x will be accepted.

This recovers Equation 1.1, except for the factor $\mathcal{A}(x/B)$ and a few previously-ignored terms in x_0. Thus, providing measurements are corrected for $\mathcal{A}(x/B)$, one has a measurement of the unpolarized decay spectrum.

2.4 First-Order Radiative Corrections

While the calculation of the muon decay spectrum seems straightforward, a complication exists: the charged particles in the decay couple to photons, so that diagrams with internal or external photon lines must be included in serious calculations. The fractional effect on the spectrum can be much larger than naive estimates of $O(\alpha)$; at $x = 0.1$, for example, the rate is increased by about 25%, and by even more at smaller x. The corrected spectrum, calculated for the V-A interaction and normalized so that the integral over x is unity, is shown in Figure 4, while the fractional effect of the first-order radiative corrections is shown in Figure 5. There is a logarithmic divergence to $-\infty$ at $x = 1$.

Because of the large size of the radiative corrections, a natural concern is the extent to which they are model-dependent. This concern is deepened by what appears to be the only explicit, published calculation of these corrections in the standard model, that of Fukuda and Sasaki.[25] They find a term proportional to $\log(m_e/M_W)$ in the radiative corrections, which causes a divergent deviation from the four-point Fermi interaction as $M_W \to \infty$. This contradicts experiment, as well as expectation, and there can be little doubt that an error exists in their work.

[25] R. Fukuda and R. Sasaki, Lett. Nuovo Cimento **10**, 17 (1974).

Ross[26] and Sirlin[27] more reasonably conclude that the spectrum corrections are terms of order $\alpha(M_\mu^2/M_W^2)$, $\alpha(q^2/M_W^2)$ and $\alpha(m_e^2/M_W^2)$ (q is the momentum transfer in the diagram), which are completely negligible for any likely experiment. Elsewhere[28] Sirlin estimates the corrections to the parameters ρ and ξ as 5.8×10^{-7} and 1.0×10^{-6}, respectively. He does not, however, give values for the corrections to the other parameters, or an explicit form for the corrections.

Because corrections due to the finite mass of the W are so small for a V-A interaction, one expects to be able to do general radiative calculations in the four-point Fermi interaction model. This is, however, not possible for general weak-interaction Lagrangians, as Sirlin discusses:

> For S, P, T, S', P', and T' interactions in the charge-retention order the corrections are divergent, which is a reflection of the non-renormalizability of the local theory. In comparing experiments with the general four-component theory, it has become customary to describe the radiative corrections by means of the finite expressions obtained for the V, A, V', and A' interactions. The justification for this procedure is that the experimental information is consistent with pure V, A, V', and A' interactions and, therefore, terms of order $\alpha/2\pi$ times ($|C_S|^2$, $|C'_S|^2$, $|C_T|^2$, $|C'_T|^2$, $|C_P|^2$, and $|C'_P|^2$) are regarded as being of second order in the small quantities. To the extent that measurements are consistent with the existence of these interactions only, the procedure is rational to check the consistency. From a theorist's point of view, it is more satisfactory to restrict oneself to the two-component theory (in which case the corrections are finite), and attempt to ...verify the quality of the fit.

Thus, should it be determined that experiment is inconsistent with the inclusion of only vector and axial-vector couplings, the traditional formulae for radiative corrections would

[26] D. A. Ross, Nucl. Phys. **B51**, 116 (1973).

[27] A. Sirlin, Nucl. Phys. **B71**, 29 (1974).

[28] A. Sirlin, Proceedings of the TRIUMF Muon Physics/Facility Workshop, p.81 (1980).

need to be replaced before one could be said to have determined the level of deviation.

Radiative corrections to the spectrum were first studied in the late fifties.[29,30] In these early treatments, m_e was set to zero wherever this would not cause a spurious divergence. For the low positron energies with which we are concerned, the calculation by Grotch[31] is more accurate, as it does not make this approximation. The remaining approximations are that it, like the previous works, considers only single photon diagrams with a four-point Fermi interaction involving only $C'_V = C_V$ and $C'_A = C_A$, with all other $C_j = C'_j = 0$. This implies that $\rho = 3/4$, among other things.

Grotch also provides an approximate formula which he says is "good to a few percent down to $E_e = 3m_e$." The approximation is much better than this modest statement implies, as Figure 6 shows in the region below $x = 0.1$. The accuracy is even better in the region used for this η parameter measurement. Nonetheless, the exact first-order formula was used in the calculations for this experiment (since it was already computed to check the approximation). It is given below in a different notation, which allows one to calculate the spectrum as an explicit, linear function of η. This simplifies the fitting of the data to the theory, compared to Grotch's representation.

$$\frac{d\Gamma(x,\eta)}{dx} = \frac{|C_A|^2 + |C_V|^2}{2}\frac{M_\mu^5}{192\pi^3}2x(x^2 - x_0^2)^{\frac{1}{2}}[f_1(x) + \eta f_2(x)]\ ,$$

where

$$\eta = \tfrac{1}{2}\tfrac{|C_A|^2 - |C_V|^2}{|C_A|^2 + |C_V|^2}\ ,$$

$$f_1(x) = A(x) + \tfrac{e^2}{2\pi}\left[A(x)B(x) + C(x) + E(x)F(x) + G(x)H(x)\right]\ ,$$

$$f_2(x) = F(x) + \tfrac{e^2}{2\pi}\left[4A(x)E(x) + B(x)F(x) + 2H(x)\tfrac{\theta}{\sinh\theta}\right]\ ,$$

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[29] R. E. Behrends, R. J. Finkelstein and A. Sirlin, Phys. Rev. **101**, 866 (1956).

[30] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).

[31] H. Grotch, Phys. Rev. **168**, 1872 (1968).

and where

$$
\begin{aligned}
A(x) &= 3 - 2x - \tfrac{x_0^2}{x} \\
B(x) &= B_1(x) + 2B_2(x)\coth\theta \\
B_1(x) &= -2\omega + V(x)\left[\omega + \theta - Z(x) - \tfrac{e^{-\omega}}{\cosh\theta}\right] + 2\left[V(x) + \tfrac{\sinh\omega}{\sinh\theta} - 1\right]\ln\left(1 - e^{-\theta-\omega}\right) \\
&\quad +2\left[V(x) - \tfrac{\sinh\omega}{\sinh\theta} - 1\right]\ln\left(1 - e^{\theta-\omega}\right) + R(x)\left(\cosh\omega - \tfrac{2}{3}\cosh\theta\right) \\
B_2(x) &= (\theta-\omega)\ln\left(\tfrac{e^{\omega}-e^{\theta}}{e^{\omega+\theta}-1}\right) - \tfrac{\pi^2}{12} + \left[\theta + \tfrac{1}{2}\ln 2 - Z(x)\right](2\theta + \ln 2) + Z(x)\left[Z(x) - \theta\right] \\
&\quad +L\left(\tfrac{2\sinh\theta}{e^{\theta}-e^{-\omega}}\right) - L\left(\tfrac{2\sinh\theta}{e^{\omega}-e^{-\theta}}\right) + L\left(e^{-\theta-\omega}\right) - L\left(e^{\theta-\omega}\right) + L(\tanh\theta) \\
&\quad +L\left(\tfrac{1-e^{-2\theta}}{2}\right) + L\left(\tfrac{1}{1+e^{2\theta}}\right) \\
C(x) &= 4\tfrac{x_0^2}{x}\left[\tfrac{\theta}{\tanh\theta} - 1\right]\sinh^2\omega \\
E(x) &= \tfrac{1}{6}R(x) - \tfrac{\theta}{\sinh\theta} \\
F(x) &= 6\tfrac{x_0}{x}(1-x) \\
G(x) &= \tfrac{5}{3}\left[(2\cosh\theta + \cosh\omega)\tfrac{\theta}{\sinh\theta} - 2\right] \\
H(x) &= \tfrac{x_0^2}{x}(\cosh\omega - \cosh\theta)^2 \\
L(t) &= \textstyle\int_0^y \left[\ln(1-t')/t'\right] dt' \\
R(x) &= \frac{\omega\sinh\omega - \frac{2}{3}\theta\sinh\theta}{(\cosh\omega - \frac{2}{3}\cosh\theta)^2 - \frac{1}{9}} \\
V(x) &= 2\theta\coth\theta \\
Z(x) &= \ln\left(1 + e^{2\theta}\right) .
\end{aligned}
$$

These expressions use two new variables:

$$\cosh\theta = x \quad \text{and} \quad \omega = \ln\left(\frac{M_\mu}{m_e}\right) .$$

2.5 Higher-Order Radiative Corrections

Because of the size of the first-order radiative corrections, it is not obvious that higher-order diagrams will not also have significant effects. Indeed, at the spectrum endpoint, higher-order terms constitute an infinite correction, eliminating the logarithmic, infrared-photon divergence in the first-order terms.[32,33,34] While theorists agree that these infrared corrections are to be handled by exponentiating terms in the first-order correction, they disagree on the specifics of the calculation. Fortunately, these discrepancies are not very large and have no significant effect on this measurement of η. The fractional effect of the exponentiation, according to the prescription of Ross, is shown in Figure 7.

Higher-order terms are much more difficult to calculate away from the endpoint since the photons are no longer necessarily soft; a comprehensive calculation has not been done. However, a calculation[35] for $x < 0.1$ implies that the effect is small for this measurement of η. Though the relative effect on the spectrum becomes substantial at lower energies, the spectrum is increased by only about 0.09% at $x_e = 0.1$. This corresponds to an effect on a measurement of η of about 0.007—were the measurement to be done with a single data point at $x_e = 0.1$ and an absolute normalization (the effect on the spectrum shape in no way simulates a value of η). The effective value of η fit with this procedure is almost proportional to $1/x$ at $x = 0.1$.

The conclusion is that it is adequate to use only the first-order radiative corrections for this particular measurement of the η parameter. Measurements of higher accuracy ($\Delta\eta < 0.01$) or which use very low-energy positron data ($x < 0.1$) may need to include higher-order terms.

2.6 Radiative Decay

Closely related to radiative corrections is radiative decay, $\mu^+ \to e^+\nu_e\overline{\nu}_\mu\gamma$. The literature contains references to this process as having a relative branching ratio of about 10^{-4}, which is surprisingly small in view of the size of the radiative corrections, even though virtual photon processes add to the latter. The resolution of this paradox is that only decays with

[32] L. Mattson, Nucl. Phys. **12B**, 647 (1969).

[33] K. A. Edin, K. E. Eriksson, V. Gerdjikov and L. Matsson, Physica Scripta **2**, 237 (1970).

[34] D. A. Ross, Nuovo Cim. **10A**, N. 3, 475 (1972).

[35] A. V. Kuznetsov and N. V. Mikheev, Sov. J. Nucl. Phys. **31**(1), (1980).

γ's more energetic than about 45 MeV are included in this 10^{-4} figure; this is presumably because actual measurements of the photon energy spectrum are done at high energies to avoid bremsstrahlung contamination.

Calculations of radiative decay have been done for both the V-A interaction[36,37] and general Lorentz structures.[38] Results show that if the arbitrary low-energy cut is made at $2m_e$, the branching ratio rises to 4.9% and that, for low positron energies, the radiative decay constitutes an even larger relative rate. By way of example, one may calculate the relative cross section at $x = 0.1$ to be about 27% for $E_\gamma > 2m_e$. This is about what one expects from the size of the radiative corrections and is large enough to indicate that the photons from radiative decay are not necessarily a negligible problem. The process, as it relates to this experiment, will be discussed in Section 5.1.6.

2.7 Effect of ρ on the Spectrum

Let us recall the uncorrected formula for unpolarized muon decay:

$$\frac{d\Gamma^{(0)}(x)}{dx} = -\frac{A}{16}\frac{M_\mu^5}{192\pi^3}\left(1+\frac{m_e^2}{M_\mu^2}\right)^4 2x\sqrt{x^2-x_0^2}\left[6(1-x)+\frac{4}{3}\rho\left(4x-3-\frac{x_0^2}{x}\right)+6\eta x_0\frac{(1-x)}{x}\right] .$$

Since ρ is dominant in determining the spectrum shape, there is a very real danger that uncertainty in its value will dominate the other errors in measuring η (or, to phrase it differently, there is a tendency for any measurement of the unpolarized spectrum to become a measurement of ρ). Neglecting x_0 terms and using the experimental fact that $\rho \approx \frac{3}{4}$, we estimate the fractional uncertainty in the spectrum relating to ρ as

$$\left.\frac{\frac{\partial(d\Gamma(x)/dx)}{\partial\rho}}{d\Gamma(x)/dx}\Delta\rho\right|_{\rho=\frac{3}{4}} = -\left(\frac{4}{3}\right)\left(\frac{3-4x}{3-2x}\right)\Delta\rho .$$

This is encouraging since the *shape* of the spectrum at small enough x becomes independent of ρ, but it is also a warning that amplitude normalization points should not be concentrated

[36]T. Kinoshita and A. Sirlin, Phys. Rev. Lett. **2**, 177 (1959).

[37]S. G. Eckstein and R. H. Pratt, Ann. Phys. **8**, 297 (1959).

[38]C. Fronsdal and H. Überall, Phys. Rev. **113**, 654 (1959).

at high values of x, to minimize the effect of ρ.

Fortunately, the present knowledge of ρ is good, compared to the accuracy likely to be achieved in η; the current world average is[39]

$$\rho = 0.7518 \pm 0.0026 \; .$$

Inserting $\Delta\rho = 0.0026$ in the above equation, the fractional uncertainty in the spectrum shape for $0.1 \leq x \leq 0.5$ is less than 0.15%. When radiative corrections to the spectrum are considered, this is reduced to 0.10%.

In order to estimate the uncertainty in η resulting from that in ρ, an idealized experiment will be considered in which only two points on the spectrum are measured, x_1 and x_2. The radiative corrections can be ignored for this purpose since they reduce the spectrum sensitivity to η and ρ similarly, and one finds

$$\Delta\eta = \left| -\frac{4}{3} \cdot \frac{\Delta\rho}{x_0} \cdot \frac{x_1 x_2 (x_2 - x_1)}{(3 - 2x_2)(1 - x_1)x_2 - (3 - 2x_1)(1 - x_2)x_1} \right| \; .$$

If one uses $x_1 = 0.2$, $x_2 = 0.5$ and $\Delta\rho = 0.0026$, then $\Delta\eta = 0.02$. This does not constitute the limit of accuracy in this experiment, then, though it is not insignificant. However, were one to try to normalize the spectrum using mostly the region near $x = 1$, the conclusion would be different: using $x_1 = 0.2$, $x_2 = 1.0$ yields $\Delta\eta = 0.07$. Thus, one must be careful to not rely upon the "cheap" amplitude normalization available near the endpoint.

[39] G. P. Yost et al., Review of Particle Properties, Phys. Lett. **204B**, 1 (1988).