Simple Explanation of the Canonical Momentum Effect in Solenoids.

R. Baartman

March 30, 1998

1 Introduction

Whenever a beam of charged particles is created inside a solenoidal field and subsequently extracted, there is a loss in beam quality. Conversely, a beam not created in this way suffers the same loss when injected axially into a solenoid. This is a phenomenon of very common occurrence in beam physics. Examples are when beam exits an ECR ion source, when a neutral beam is ionized inside a charge-exchange cell which has a solenoidal field (e.g. for polarized sources), when a beam is injected along the axis of a cyclotron, and when a secondary beam production target is placed inside a solenoid. Especially because of the last example, it is important that experimenters understand the effect as well.

2 Canonical Momentum Derivation

The canonical momentum of a particle of charge q is

$$\vec{P} = \vec{p} + q\vec{A}, \tag{1}$$

where $\vec{p}$ is the usual momentum, and $\vec{A}$ is the magnetic field's vector potential. In a region with no electric field the Hamiltonian is independent of position and so the canonical momentum is conserved. If a particle originates inside a magnetic field and travels to a point well outside of it, it receives a change of (ordinary) momentum

$$\Delta\vec{p} = q\vec{A}. \tag{2}$$

For a solenoid with magnetic field B_0, $\vec{A}$ has only a component in the azimuthal (θ) direction

$$A_\theta = \frac{r}{2} B_0. \tag{3}$$

So on exiting the solenoid at a radius r, the particle gets an azimuthal kick:

$$\phi = \frac{\Delta p_\theta}{p_z} = \frac{qr}{2p_z} B_0 \tag{4}$$

It is often convenient to express the momentum as a 'rigidity' $B\rho = p/q$. Then the angular kick is expressed very simply as

$$\phi = \frac{r}{2\rho}, \tag{5}$$

where ρ is the radius of curvature that the particle would have if it were travelling with momentum p_z orthogonal to the field B_0.

3 'Gaussian' Derivation

For those who find arguments based on Hamiltonian dynamics unconvincing, here is an alternative which places the blame squarely on the Maxwell equation $\nabla \cdot \vec{B} = 0$.

Take as Gaussian surface a truncated square circular cylinder of radius r coaxial with the solenoid. One end is well inside the solenoid, where the field is B_0 in the axial direction, and the other is well outside, where the field is zero. Then over this surface,

$$\oint \vec{B} \cdot d\vec{S} = 0 \tag{6}$$

so

$$\pi r^2 B_0 = 2\pi r \int B_r dz. \tag{7}$$

Simply put, all of the flux leaving the cylinder 'cap' inside the solenoid must have come in through the sides of the cylinder. This means that there is an irreducible amount of integrated radial magnetic field through the fringe field region:

$$\int B_r dz = B_0 r/2. \tag{8}$$

The effect in the fringe field on a particle of charge q travelling with axial velocity component $v_z = dz/dt$ is to kick it sideways with a force

$$F_\theta = q v_z B_r = dp_\theta/dt, \tag{9}$$

or

$$dp_\theta = qB_r dz. \tag{10}$$

The integrated kick in angle is therefore

$$\phi = \int \frac{dp_\theta}{p_z} = \frac{q}{p_z} \int B_r dz = \frac{qB_0}{p_z} \frac{r}{2}, \tag{11}$$

or, since $B_0 \rho = p_z/q$, we have simply

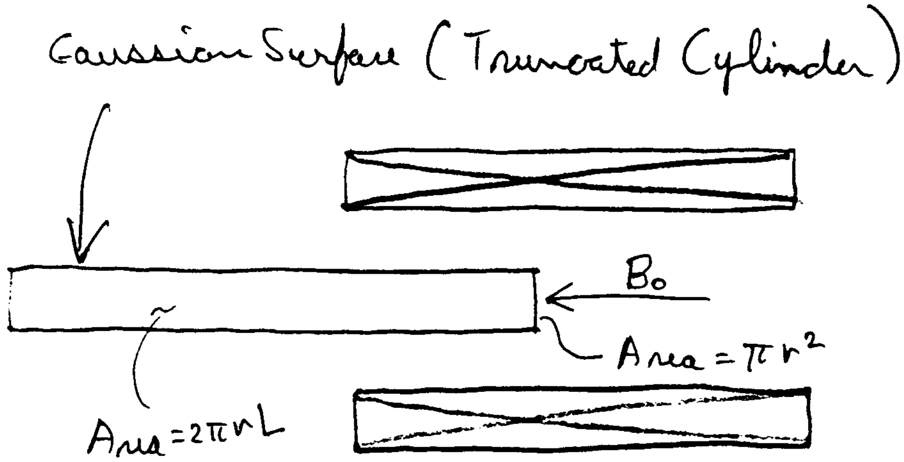
$$\phi = \frac{r}{2\rho}. \tag{12}$$

4 Example (from μ-μ collider)

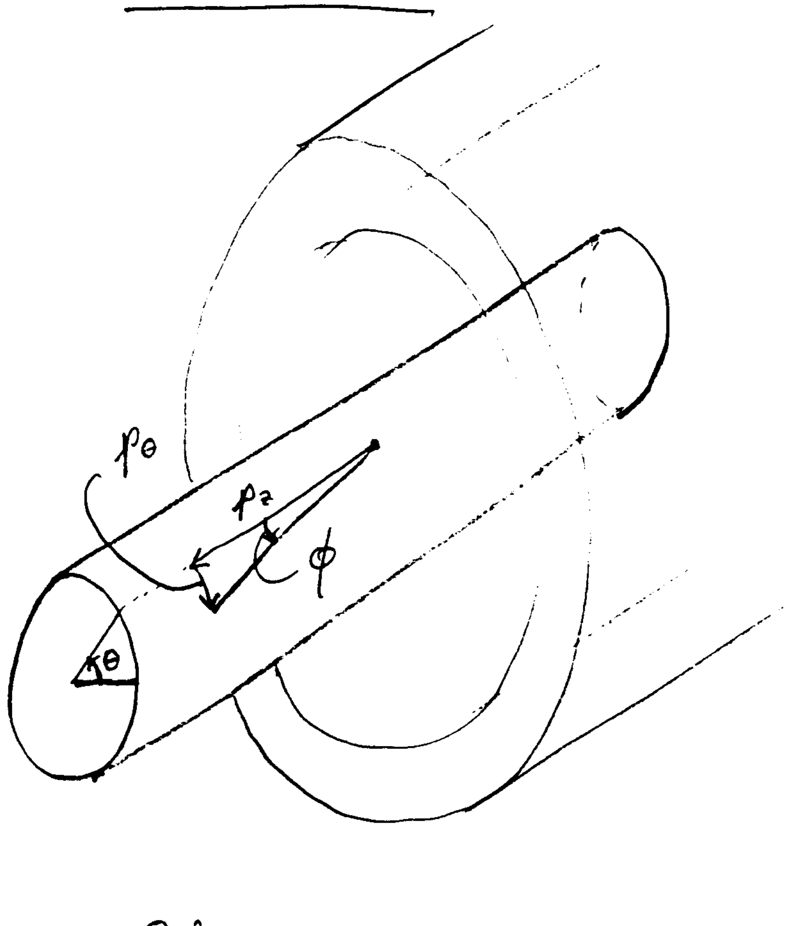
Muons of momentum 200 MeV/c are created inside a 20 T solenoid field and extracted along its axis through an aperture of radius 5 mm. $B\rho = 200\,\mathrm{MV}/3 \times 10^8\,\mathrm{m/s} = 0.67\,\mathrm{T\text{-}m}$, and so $\rho = B\rho/B = 33\,\mathrm{mm}$. The 'spray' angle of exiting particles is $\phi = 5\,\mathrm{mm}/(2 \times 33\,\mathrm{mm}) =$ 75 mrad, resulting in an emittance of (phase space area $\div\pi = r\phi$=) 375 mm-mrad.

5 Conclusion

It is important to understand that the only way to reduce the emittance growth $\epsilon_{\mathrm{solenoid}} = r^2/(2\rho)$ upon exiting axially from a solenoid is to reduce the aperture r. Yes, the fringe field is at fault, but no amount of sweating over its shape is going to improve it. Does this violate Liouville's theorem? No. A zero emittance beam born inside the solenoid still has a zero 4-dimensional phase space volume on extraction, but the $x - p_x$ and $y - p_y$ projections are not zero. Instead, the $x - p_y$ and $y - p_x$ projections are zero. These couplings cannot be corrected except by re-injecting into another solenoidal field. In effect, when extracting from a solenoid the beam comes out spinning about the axis, and however the beam is subsequently manipulated, this angular momentum cannot be made to disappear.



Side view



Oblique